Abstract

We present a mathematical model for geometric deep learning based upon a scattering transform defined over manifolds, which generalizes the wavelet scattering transform of Mallat. This geometric scattering transform is (locally) invariant to isometry group actions, and we conjecture that it is stable to actions of the diffeomorphism group.

1 Introduction

Convolutional neural networks (CNNs) are revolutionizing imaging science for two and three dimensional images over Euclidean domains. However, many images, and more generally data sets, are intrinsically non-Euclidean and are better modeled through other mathematical structures, such as graphs or manifolds. This has led to the development of geometric deep learning [1, 2] (and references therein), which refers to a body of research that aims to translate the principles of CNNs to these non-Euclidean structures. In the process, various challenges have arisen, including how to define such networks, how to train them efficiently, and how to analyze their mathematical properties. In this extended abstract we focus primarily on the last question in a relatively general context, akin to several existing methods, which illustrates the fundamental properties of such networks.

We present a geometric version of the scattering transform (Figure 1), which is similar to the one introduced in [3], but here is defined over manifolds instead of Euclidean space. The Euclidean scattering transform can be, on the one hand, thought of as a mathematical model for standard CNNs, but has, on the other hand, obtained state of the art or near state of the art empirical results in computer vision [4, 5, 6], audio signal processing [7, 8, 9], and even quantum chemistry [10, 11]. In Section 3 we define the geometric scattering transform, and provide results showing it encodes localized isometry invariant descriptions of signal data defined on a manifold. These

Figure 1: The geometric scattering transform. Black: Equivariant intermediate layers. Blue: Invariant output coefficients at each layer.
results generalize local translation and rotation invariance on Euclidean domains. The
underlying spectral integral operators that are the foundation of the geometric scattering
transform are presented in Section 2. In Section 4, we discuss the stability of the geometric
scattering transform to the action of diffeomorphisms. We provide a framework in which
to quantify how much a diffeomorphism differs from being an isometry, provide results
proving that individual spectral integral operators are Lipschitz stable to diffeomorphism
actions within this framework, and conjecture that these results can be extended to the
geometric scattering transform. Finally, we provide a short conclusion in Section 5.

1.1 Notation
Let \( M \) be a smooth, compact, and connected, \( d \)-dimensional Riemannian manifold without
boundary contained in \( \mathbb{R}^d \). Let \( r : M \times M \rightarrow \mathbb{R} \) denote the geodesic distance between
two points, and let \( \Delta \) be the Laplace-Beltrami operator on \( M \). The eigenfunctions and non-
unique eigenvalues of \(-\Delta\) are denoted by \( \varphi_k \) and \( \lambda_k \), respectively. Since \( M \) is compact, the
spectrum of \(-\Delta\) is countable and we may assume that \( \{\varphi_k\}_{k \in \mathbb{N}} \) forms an orthonormal basis
for \( L^2(M) \). The set of unique eigenvalues of \(-\Delta\) is denoted by \( \Lambda \), and for \( \lambda \in \Lambda \) we let \( m(\lambda) \)
and \( E_\lambda \) denote the corresponding multiplicities and eigenspaces. For a diffeomorphism \( \zeta : M \rightarrow M \), we let \( V_{\zeta} \) be the operator \( V_{\zeta} f(x) = f(\zeta^{-1}(x)) \), and let \( \|\zeta\|_\infty = \sup_{x \in M} r(x, \zeta(x)) \).

2 Spectral Integral Operators
For a smooth function \( \eta \), we define a spectral kernel \( K_\eta \) by
\[
K_\eta(x, y) = \sum_{k \in \mathbb{N}} \eta(\lambda_k) \varphi_k(x) \overline{\varphi_k(y)}
\]
and refer to the operator \( T_\eta \), with kernel \( K_\eta \), as a spectral integral operator. It can be verified
that
\[
T_\eta f(x) = \sum_{k \in \mathbb{N}} \eta(\lambda_k) \langle f, \varphi_k \rangle \varphi_k(x).
\]

Since \( \{\varphi_k\}_{k \in \mathbb{N}} \) is an orthonormal basis for \( L^2(M) \), it follows that
\[
\|T_\eta f\|_2^2 = \sum_{k \in \mathbb{N}} |\eta(\lambda_k)|^2 |\langle f, \varphi_k \rangle|^2.
\]

Therefore, if \( \|\eta\|_\infty \leq 1 \), then \( T \) is a nonexpansive operator on \( L^2(M) \). Operators of this
form are analogous to convolution operators defined on \( \mathbb{R}^d \) since like the latter they are
diagonalized in the Fourier basis. To further emphasize this connection, we note the
following theorem which shows that spectral integral operators are equivariant with respect
to isometries.

**Theorem 2.1.** Let \( T_\eta \) be a spectral integral operator. Then, if \( \zeta \) is an isometry,
\[
T_\eta V_{\zeta} f = V_{\zeta} T_\eta f
\]
for all \( f \in L^2(M) \).

We will consider frame operators that are constructed using a countable family of spectral
integral operators. In particular, we assume that we have a low-pass filter \( g \) and a family of
high-pass filters \( \{h_\gamma\}_{\gamma \in \Gamma} \) which satisfy a Littlewood-Paley type condition
\[
A \leq m(\lambda) \left[ |g(\lambda)|^2 + \sum_{\gamma \in \Gamma} |h_\gamma(\lambda)|^2 \right] \leq B, \quad \forall \lambda \in \Lambda \tag{1}
\]
for some \( 0 < A \leq B \). A frame analysis operator is then defined by
\[
\Phi f = \{ T_g f, T_{h_\gamma} f : \gamma \in \Gamma \},
\]
where \( T_g \) and \( T_{h_\gamma} \) are the spectral integral operators corresponding to \( \eta = g \) and \( \eta = h_\gamma \),
respectively.
Proposition 2.2. Under the Littlewood-Paley condition \( [1] \), \( \Phi \) is a bounded operator from \( L^2(M) \) to \( \ell^2(L^2(M)) \) and \( A \| f \|_2^2 \leq \| \Phi f \|_2^2 := \| T_{\gamma} f \|_2^2 + \sum_{\gamma \in \Gamma} \| T_{b_{\gamma}} f \|_2^2 \leq B \| f \|_2^2 \) for all \( f \in L^2(M) \). In particular, if \( A = B = 1 \), then \( \Phi \) is an isometry.

Remark 2.3. In [13], R. Coifman and M. Maggioni used the heat semigroup to construct a class filters such that the high-pass filters \( \{ h_j \}_{j \in \mathbb{Z}} \) form a wavelet frame and a low-pass filter \( g_j \) is chosen so that \( \{ g_j, h_j : j \geq 1 \} \) satisfies \([4]\). This construction can be generalized to the manifold setting after making suitable adjustments to account for the multiplicities of the eigenvalues.

3 The Geometric Scattering Transform

The geometric scattering transform is a nonlinear operator constructed through an alternating cascade of spectral integral operators and nonlinearities. Let \( M : L^2(M) \rightarrow L^2(M) \) be the modulus operator, \( Mf(x) = |f(x)| \), and for each \( \gamma \in \Gamma \), we let \( U_{\gamma}f(x) = MT_{b_{\gamma}}f(x) = |T_{b_{\gamma}}f(x)| \). We define an operator \( U : L^2(M) \rightarrow \ell^2(L^2(M)) \), called the one-step scattering propagator, by

\[
Uf = \{ T_{g_{\gamma}}, U_{\gamma}f : \gamma \in \Gamma \}.
\]

The m-step scattering propagator is constructed by iteratively applying the one-step scattering propagator. For \( m \geq 1 \), let \( \Gamma_m \) be the set of all paths of the form \( \tilde{\gamma} = (\gamma_1, \ldots, \gamma_m) \). Let \( \Gamma_0 \) denote the empty set, and let \( \Gamma_\infty = \bigcup_{m=1}^{\infty} \Gamma_m \) denote the set of a all finite paths. For \( \tilde{\gamma} \in \Gamma_m \), let

\[
U_{\tilde{\gamma}}f(x) = U_{\gamma_m} \cdots U_{\gamma_1}f(x), \quad \tilde{\gamma} = (\gamma_1, \ldots, \gamma_m),
\]

and for \( \mathcal{P} \subset \Gamma_\infty \), we define \( U[\mathcal{P}]f \) as the collection of all path propagators with paths in \( \mathcal{P} \),

\[
U[\mathcal{P}]f = \{ U_{\tilde{\gamma}}f : \tilde{\gamma} \in \mathcal{P} \}.
\]

The scattering transform \( S_{\tilde{\gamma}} \) over a path \( \tilde{\gamma} \in \Gamma_\infty \) is defined as the integration of \( U_{\tilde{\gamma}} \) against the low-pass integral operator \( T_{g_{\gamma}} \), i.e.

\[
S_{\tilde{\gamma}}f(x) = T_{g_{\gamma}}U_{\tilde{\gamma}}f(x).
\]

Analogously to \( U[\mathcal{P}] \), we define

\[
S[\mathcal{P}]f = \{ S_{\tilde{\gamma}}f : \tilde{\gamma} \in \mathcal{P} \}.
\]

The operator \( S[\Gamma_{\infty}] : L^2(M) \rightarrow \ell^2(L^2(M)) \) is referred to as the scattering transform. The following proposition shows that \( S[\Gamma_{\infty}] \) is nonexpansive.

Proposition 3.1. If the Littlewood-Paley condition \([1]\) holds, then

\[
\| S[\Gamma_{\infty}]f_1 - S[\Gamma_{\infty}]f_2 \|_{2,2} \leq \| f_1 - f_2 \|, \quad \forall f_1, f_2 \in L^2(M).
\]

The scattering transform is invariant to the action of the isometry group on the inputted signal \( f \) up to a factor that depends upon the decay of the low-pass spectral function \( g \). If the low-pass spectral function \( g \) is rapidly decaying and satisfies \( |g(\lambda)| \leq Ce^{-t\lambda} \) for some constant \( C \) and \( t > 0 \) (e.g., the heat kernel), then the following theorem establishes isometric invariance up to the scale \( t^d \).

Theorem 3.2. Let \( \xi \) be an isometry. If the Littlewood-Paley condition \([1]\) holds and \( |g(\lambda)| \leq Ce^{-t\lambda} \) for some constant \( C \) and \( t > 0 \), then there exists a constant \( C(M) < \infty \), such that

\[
\| S[\Gamma_{\infty}]f - S[\Gamma_{\infty}]V_{\xi}f \|_{2,2} \leq C(M)t^{-d} \| \xi \|_{\infty} \| U[\Gamma_{\infty}]f \|_{2,2} \quad \forall f \in L^2(M).
\]


4 Stability to Diffeomorphisms

As stated in Theorem 2.1, spectral integral operators are equivariant to the action of isometries. This fact is crucial to proving Theorem 3.2 because it allows us to estimate

\[ \| S[\Gamma_\infty]f - V_\zeta S[\Gamma_\infty]f \|_{2,2} \]  

instead of

\[ \| S[\Gamma_\infty]f - S[\Gamma_\infty] V_\zeta f \|_{2,2}. \]  

In [3], it is shown that the Euclidean scattering transform \( S_{Euc} \) is stable to the action of certain diffeomorphisms which are close to being translations. A key step in the proof is a bound on the commutator norm \( \| [S_{Euc}[\Gamma_\infty], V_\zeta] \| \), which then allows the author to bound a quantity analogous to (2) instead of bounding (3) directly. This motivates us to study the commutator of spectral integral operators with \( V_\zeta \) for diffeomorphisms which are close to being isometries.

For technical reasons, we will assume that \( \mathcal{M} \) is two-point homogeneous, that is, for any two pairs of points, \( (x_1, x_2), (y_1, y_2) \) such that \( r(x_1, x_2) = r(y_1, y_2) \), there exists an isometry \( \zeta : \mathcal{M} \to \mathcal{M} \) such that \( \zeta(x_1) = y_1 \) and \( \zeta(x_2) = y_2 \). In order to quantify how far a diffeomorphism \( \zeta \) differs from being an isometry we will consider two quantities:

\[ A_1(\zeta) = \sup_{x,y \in \mathcal{M}} \left| \frac{r(\zeta(x), \zeta(y)) - r(x, y)}{r(x, y)} \right|, \]

and

\[ A_2(\zeta) = \sup_{x \in \mathcal{M}} | |\det[D\zeta(x)]|| - 1|. \]

We let \( A(\zeta) = \max \{ A_1(\zeta), A_2(\zeta) \} \) and note that if \( \zeta \) is an isometry, then \( A(\zeta) = 0 \).

**Theorem 4.1.** Assume that \( \mathcal{M} \) is two-point homogeneous, and let \( T_\eta \) be a spectral integral operator. Then there exists a constant \( C(\mathcal{M}) > 0 \) such that for any diffeomorphism \( \zeta : \mathcal{M} \to \mathcal{M} \),

\[ \| [T, V_\zeta] \| \leq C(\mathcal{M}) A(\zeta) B(\eta) \]  

where

\[ B(\eta) = \max \left\{ \sum_{k \in \mathbb{N}} \eta(\lambda_k) \lambda_k^{(d-1)/4}, \left( \sum_{k \in \mathbb{N}} \eta(\lambda_k^2) \right)^{1/2} \right\}. \]

**Remark 4.2.** We conjecture that when \( \Phi \) is constructed to be a wavelet frame as in Remark 2.3, we can use (4) to prove a bound on \( \| S[\Gamma_\infty], V_\zeta \| \) in terms of \( A(\zeta) \). If true, this result would allow us to show that the scattering transform is stable to diffeomorphisms using methods analogous to the ones in [3].

5 Conclusion

We have presented a path towards understanding the mathematical properties of geometric deep networks through the notion of the geometric scattering transform. Recently, related analyses for graphs have been presented in [14][15][16]. We remark that the analysis proposed here applies to compact Riemannian manifolds of arbitrary dimension, thus providing a road map that goes beyond 2D surface or 3D shape matching. Looking ahead, such an approach naturally lends itself to research avenues that synthesize geometric deep learning and geometric data analysis (e.g., manifold learning [17][18][19]), which in turn has the potential to bridge the graph and manifold theories for geometric deep networks.

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References


