Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 01: Fourier, heat diffusion, and shift-invariant operators

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1 Sparse Representations

Chapter 1 of A Wavelet Tour of Signal Processing [1].

Exercise 1. Read Chapter 1 (*Sparse Representations*) of *A Wavelet Tour of Signal Processing*. It gives a nice overview of the book and will give you a good perspective on computational harmonic analysis heading into the course.

Exercise 2. Read the appendices in *A Wavelet Tour of Signal Processing*, as we will not cover these in class. We will immediately need some of the material contained in them.

Remark 1.1. The integral we use in this course will be the Lebesgue integral, which is usually taught in a first year graduate course in real analysis. However, if these are unfamiliar to you, you may replace most if not all of the results with Riemann integrals from Calculus and assume that the generic functions f, g, h, etc. are Schwartz class functions. For more details on the Schwartz class and Fourier integrals, see [2].

2 The Fourier Kingdom

Chapter 2 of A Wavelet Tour of Signal Processing [1].

2.1 Linear time-invariant filtering

Section 2.1 of A Wavelet Tour of Signal Processing [1].

Fourier analysis originates with the work of Joseph Fourier, who was studying the heat equation:

$$\partial_t F = \Delta F$$
$$F(u,0) = f(u)$$

Where $F : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R}$. This is a linear partial differential equation. In order to solve it, it helps to think about linear algebra. Suppose $\mathbf{A}$ is an $n \times n$ real valued, symmetric matrix, which maps vectors $x \in \mathbb{R}^n$ to other vectors $\mathbf{A}x \in \mathbb{R}^n$. Then from the spectral theorem, we know that $\mathbf{A}$ has a complete set of orthonormal eigenvectors, $v_1, \ldots, v_n$, such that

$$\mathbf{A}v_k = \lambda_k v_k$$

for some $\lambda_k \in \mathbb{R}$. Since $\{v_k\}_{k \leq n}$ forms an ONB, it allows us to write, for any $x \in \mathbb{R}^n$,

$$x = \sum_{k=1}^{n} \langle x, v_k \rangle v_k \,,$$

which in turn makes evaluating $\mathbf{A}x$ very easy:

$$\mathbf{A}x = \sum_{k=1}^{n} \langle x, v_k \rangle \mathbf{A}v_k = \sum_{k=1}^{n} \lambda_k \langle x, v_k \rangle v_k \,.$$

Let us now try to apply the same ideas to the Laplacian, Δ. We may ask, what are the eigenfunctions of the Laplacian? If we consider complex valued functions, one can verify that

$$\Delta e^{i\omega \cdot u} = -|\omega|^2 e^{i\omega \cdot u}$$

for any $\omega \in \mathbb{R}^d$. Thus the function $e_\omega(u) = e^{i\omega \cdot u}$ is an eigenfunction of Δ for any ω. Let us formally define

$$\widehat{f}(\omega) = \langle f, e_\omega \rangle = \int_{\mathbb{R}^d} f(u) e^{-i\omega \cdot u} \, du \,.$$

This will be what we call the Fourier transform, but right now we see it as an analogue of basis coefficients in an ONB. Following the analogy, we may then be tempted to write:

$$f(u) = \int_{\mathbb{R}^d} \langle f, e_\omega \rangle e_\omega(u) \, d\omega = \int_{\mathbb{R}^d} \widehat{f}(\omega) e^{i\omega \cdot u} \, d\omega \,.$$

We this in hand, we then propose

$$F(u,t) = \int_{\mathbb{R}^d} e^{-|\omega|^2 t} \widehat{f}(\omega) e^{i\omega \cdot u} \, d\omega \,,$$

as the solution to the heat equation. One can verify that, formally, F indeed is the solution. Fourier analysis was then born by trying to understanding when all of this makes mathematical sense.

The reason Fourier analysis is used so often in signal processing, is that it turns out this analysis is not useful for just the Laplacian operator. In fact the Laplacian is just an example of a more general class operators, called *shift invariant* operators. Let us now work over $\mathbb{R}$ instead of $\mathbb{R}^d$; we will use t to denote a value in $\mathbb{R}$, since it is often useful to think of it as time. Let $f_\tau(t) = f(t - \tau)$ be the translation of f by τ; if t is time, then this is a time delay by τ. An operator L is shift invariant if it commutes with the time delay of any function,

$$(Lf_\tau)(t) = (Lf)(t - \tau)$$

As we shall see all linear, continuous shift invariant operators L are diagonalized by the complex exponentials $e_\omega(t) = e^{i\omega t}$. To see this, recall the *convolution* of two functions f, g:

$$f * g(t) = \int_{\mathbb{R}} f(u)g(t-u)\, du$$

Now let $\delta(t)$ be a Dirac (centered at zero), and $\delta_u(t) = \delta(t-u)$ be a Dirac centered at u. By definition this means $f * \delta(t) = f(t)$. We have:

$$f(t) = f * \delta(t) = \int_{\mathbb{R}} f(u)\delta(t-u)\, du = \int_{\mathbb{R}} f(u)\delta_u(t)\, du$$

Since L is continuous and linear,

$$Lf(t) = \int_{\mathbb{R}} f(u) L\delta_u(t)\, du$$

Let h be the impulse response of L, defined as

$$h(t) = L\delta(t)$$

Since L is shift invariant, we have

$$L\delta_u(t) = h(t-u)$$

and therefore

$$Lf(t) = \int_{\mathbb{R}} f(u)h(t-u)\, du = f * h(t) = h * f(t)$$

Thus *every* continuous, linear shift invariant operator is equivalent to a convolution with an impulse response h.

We can now use this fact to show our original goal, which was that the complex exponential functions $e_\omega(t) = e^{i\omega t}$ diagonalize L. This will in turn motivate the study of Fourier integrals. We have:

$$Le_\omega(t) = \int_{\mathbb{R}} h(u)e^{i\omega(t-u)}\, du = e^{it\omega} \underbrace{\int_{\mathbb{R}} h(u)e^{-i\omega u}\, du}_{\widehat{h}(\omega)} = \widehat{h}(\omega)e_\omega(t).$$

Thus $e_\omega(t)$ is an eigenfunction of L with eigenvalue $\widehat{h}(\omega)$, if $\widehat{h}(\omega)$ exists. The value $\widehat{h}(\omega)$ is the *Fourier transform* of h at the frequency ω. Since the functions $e_\omega(t) = e^{i\omega t}$ are the eigenfunctions of shift invariant operators, we would like to decompose any function f as a sum or integral of these functions. This will then allow us to write Lf directly in terms of the eigenvalues of L (as you do in linear algebra when you are able to diagonalize a matrix/operator on a finite dimensional vector space). We'll try to undestand when this is possible.

Exercise 3. Read Section 2.1 of *A Wavelet Tour of Signal Processing.*

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.