Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 02: The Fourier transform on $\mathbf{L}^1(\mathbb{R})$

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2.2 Fourier integrals

Section 2.2 of A Wavelet Tour of Signal Processing [1].

The Fourier transform is an operator $\mathcal{F}$ that maps a function $f(u)$ to another function $\widehat{f}(\omega)$, which is defined as:

$$\mathcal{F}(f)(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t}\, dt \tag{1}$$

We will start by trying to understand what restrictions we need to place on f in order for this to make sense. In particular, if f is in some well defined space of functions, we will ask, does that imply $\widehat{f}$ is in some other well defined space of functions? We will start by considering the $\mathbf{L}^p$ spaces of functions. To that end, define:

$$\mathbf{L}^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{\mathbb{R}} |f(t)|^p \, dt < +\infty \right\}, \quad 0 < p < \infty$$

The space $\mathbf{L}^p(\mathbb{R})$ is a Banach space with norm:

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p \, dt \right)^{\frac{1}{p}}$$

The space $\mathbf{L}^2(\mathbb{R})$ is special, as it is in fact a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t) g^*(t) \, dt$$

where we use $g^*(t)$ to denote the complex conjugate of $g(t)$. We also define $\mathbf{L}^\infty(\mathbb{R})$. Set:

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in \mathbb{R}} |f(t)|$$

The value $\|f\|_\infty$ is the smallest number M, $0 \leq M \leq +\infty$, such that $|f(t)| \leq M$ for almost every $t \in \mathbb{R}$; if f is continuous, it is the smallest number M such that $|f(t)| \leq M$ for all $t \in \mathbb{R}$. It thus measures whether f is bounded or not. The space $\mathbf{L}^\infty(\mathbb{R})$ is the space of bounded functions:

$$\mathbf{L}^\infty(\mathbb{R}) = \{ f : \|f\|_\infty < +\infty \}$$

We then have:

Proposition 2.1. *If* $f \in \mathbf{L}^1(\mathbb{R})$*, then* $\widehat{f} \in \mathbf{L}^\infty(\mathbb{R})$*.*

Proof. Suppose $f \in \mathbf{L}^1(\mathbb{R})$. We have:

$$|\widehat{f}(\omega)| = \left| \int_{\mathbb{R}} f(t) e^{-i\omega t} \, dt \right| \leq \int_{\mathbb{R}} |f(t) e^{-i\omega t}| \, dt = \int_{\mathbb{R}} |f(t)| \, dt = \|f\|_1 < +\infty$$

□

Proposition 2.1 shows that $\mathcal{F} : \mathbf{L}^1(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R})$ is a well defined map using the definition (1). Later on we will extend the Fourier transform to other $\mathbf{L}^p$ spaces for $1 \leq p \leq 2$, with particular interest in $\mathbf{L}^2(\mathbb{R})$. For now recall from Section 2.1 that we would like to write $f(t)$ in terms of $\widehat{f}(\omega)$. This requires a Fourier inversion formula. However, the above proposition only guarantees that $\widehat{f} \in \mathbf{L}^\infty(\mathbb{R})$, which will not help with convergence issues. We thus assume that $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$ as well.

Theorem 2.2 (Fourier inversion)**.** *If* $f \in \mathbf{L}^1(\mathbb{R})$ *and* $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$ *then*

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) e^{i\omega t} \, d\omega, \quad \text{for almost every } t \in \mathbb{R} \tag{2}$$

To prove this theorem, we will need three standard results from graduate real analysis. We state them here, without proof.

Theorem 2.3. *Suppose* $\{f_n\}_{n \in \mathbb{N}}$ *converges to* f *in* $\mathbf{L}^p$*, meaning that*

$$\lim_{n \to \infty} \|f_n - f\|_p = 0$$

Then there exists a subsequency $\{f_{n_k}\}_{k \in \mathbb{N}}$ *that converges to* f *almost everywhere,*

$$\lim_{k \to \infty} f_{n_k}(t) = f(t) \quad \text{for almost every } t$$

Theorem 2.4 (Dominated Convergence Theorem)**.** *Let* $\{f_n\}_{n \in \mathbb{N}}$ *be a sequence of functions such that* $\lim_{n \to \infty} f_n = f$*. If*

$$\forall\, n \in \mathbb{N}, \quad |f_n(t)| \leq g(t) \quad \text{and} \quad \int_{\mathbb{R}} g(t) \, dt < +\infty$$

then $f \in \mathbf{L}^1(\mathbb{R})$ *and*

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(t) \, dt = \int_{\mathbb{R}} f(t) \, dt$$

Theorem 2.5 (Fubini's Theorem)**.** *Let* $f(u, t)$ *be a function of two variables* $(u, t) \in \mathbb{R}^2$*. If*

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(u, t)| \, du \right) dt < +\infty$$

then

$$\iint_{\mathbb{R}^2} f(u,t)\,du\,dt = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u,t)\,du \right) dt$$
$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u,t)\,dt \right) du$$

Proof of Theorem 2.2. Now we turn to the proof. Plugging in the formula of $\widehat{f}(\omega)$ into the right hand side of (2) yields

$$\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) e^{i\omega t} = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u) e^{i\omega(t-u)}\,du \right) d\omega$$

However we cannot apply Fubini directly because the function $F(u,\omega) = f(u)e^{i\omega(t-u)}$ is not integrable in $\mathbb{R}^2$. Therefore we instead consider the following integral:

$$I_\varepsilon(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(u) e^{-\varepsilon^2\omega^2/4} e^{i\omega(t-u)}\,du \right) d\omega$$

The Gaussian yields a new integrand $F_\varepsilon(u,\omega) = f(u)e^{-\varepsilon^2\omega^2/4}e^{i\omega(t-u)}$ which is integrable on $\mathbb{R}^2$, and for which $\lim_{\varepsilon\to 0} F_\varepsilon = F$. We can thus apply the Fubini theorem to $I_\varepsilon(t)$; we do so in two ways. For the first, we integrate with respect to u, giving:

$$I_\varepsilon(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) e^{-\varepsilon^2\omega^2/4} e^{i\omega t}\,d\omega.$$

Since

$$\left| \widehat{f}(\omega) e^{-\varepsilon^2\omega^2/4} e^{i\omega t} \right| \leq |\widehat{f}(\omega)|$$

and since $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$, we can apply the dominated convergence theorem to obtain:

$$\lim_{\varepsilon\to 0} I_\varepsilon(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) e^{i\omega t}\,d\omega \tag{3}$$

Now compute $I_\varepsilon(t)$ a second way by applying the Fubini theorem and integrating with respect to ω. We get that

$$I_\varepsilon(t) = \int_{\mathbb{R}} g_\varepsilon(t-u) f(u)\,du = f * g_\varepsilon(t)$$

where

$$g_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\varepsilon^2\omega^2/4} e^{ix\omega}\,d\omega$$
$$= \frac{1}{\varepsilon\sqrt{\pi}} \int_{\mathbb{R}} \frac{\varepsilon}{2\sqrt{\pi}} e^{-\varepsilon^2\omega^2/4} e^{ix\omega}\,d\omega$$
$$= \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2}$$

To get the last line, we used the fact that that the Fourier transform of $\theta(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$ is equal to $\widehat{\theta}(\xi) = e^{-\sigma^2\xi^2/2}$. This is a useful identity that you should verify yourself and then remember. Another useful identity is that $\int_{\mathbb{R}} \theta(t)\,dt = 1$. From this latter formula we deduce that

$$\int_{\mathbb{R}} g_\varepsilon(x)\,dx = 1$$

Furthermore, we notice that

$$g_\varepsilon(x) = \varepsilon^{-1} g_1(\varepsilon^{-1}x), \quad g_1(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \tag{4}$$

Thus the family $\{g_\varepsilon\}_{\varepsilon>0}$ is an approximate identity. For general approximate identities one can prove (see below):

$$\lim_{\varepsilon\to 0} \|f * g_\varepsilon - f\|_1 = 0$$

We now apply Theorem 2.3 to infer there exists a subsequence $\{f * g_{\varepsilon_k}\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ such that $\lim_{k\to\infty} f * g_{\varepsilon_k} = f$ almost everywhere. On the other hand, using (3) we have

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)e^{i\omega t}\,d\omega = \lim_{k\to\infty} I_{\varepsilon_k}(t) = \lim_{k\to\infty} f * g_{\varepsilon_k}(t) = f(t) \quad \text{for almost every } t$$

thus completing the proof. □

To complete the above proof of Theorem 2.2, we introduce the notion of an *approximate identity*. A family of functions $\{k_\lambda\}_{\lambda>0}$ is an approximate identity if:

1. Normalized: $\int_{\mathbb{R}} k_\lambda(t)\,dt = 1$ for every $\lambda > 0$
2. $\mathbf{L}^1$-boundedness: $\sup_{\lambda>0} \|k_\lambda\|_1 < \infty$
3. $\mathbf{L}^1$-concentration: For every $\delta > 0$,

$$\lim_{\lambda\to 0}\int_{|t|\geq\delta} |k_\lambda(t)|\,dt = 0$$

One can verify that the family of Gaussian functions $\{g_\varepsilon\}_{\varepsilon>0}$ from (4) is an approximate identity. More generally, one often constructs an approximate identity by dilating a single function $k \in \mathbf{L}^1(\mathbb{R})$ satisfying $\|k\|_1 = 1$, that is by setting $k_\lambda(t) = \lambda^{-1}k(\lambda^{-1}t)$. The following theorem is useful on its own, and completes the proof of Theorem 2.2.

Theorem 2.6. *Let $\{k_\lambda\}_{\lambda>0}$ be an approximate identity. Then*

$$\forall\, f \in \mathbf{L}^1(\mathbb{R}), \quad \lim_{\lambda\to 0} \|f - f * k_\lambda\|_1 = 0$$

To prove Theorem 2.6 we will need the following standard result from real analysis.

Theorem 2.7 ($\mathbf{L}^p$ continuity)**.** *Let* $1 \le p < \infty$. *If* $f \in \mathbf{L}^p(\mathbb{R})$, *then*

$$\lim_{\tau \to 0} \|f - f_\tau\|_p = 0$$

where we recall $f_\tau(t) = f(t - \tau)$. *By the definition of limit, this means for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|\tau| < \delta \Longrightarrow \|f - f_\tau\|_p < \varepsilon$$

Proof of Theorem 2.6. Since $f, k_\lambda \in \mathbf{L}^1(\mathbb{R})$, one can show that $\|f * k_\lambda\|_1 \le \|f\|_1 \|k_\lambda\|_1 < \infty$, which means that $f * k_\lambda \in \mathbf{L}^1(\mathbb{R})$. Using the fact that $\int_{\mathbb{R}} k_\lambda = 1$ for all $\lambda > 0$, we have:

$$\begin{aligned}
\|f - f * k_\lambda\|_1 &= \int_{\mathbb{R}} |f(t) - f * k_\lambda(t)| \, dt \\
&= \int_{\mathbb{R}} \left| f(t) \int_{\mathbb{R}} k_\lambda(u) \, du - \int_{\mathbb{R}} f(t-u) k_\lambda(u) \, du \right| \, dt \\
&= \iint_{\mathbb{R}^2} |f(t) - f(t-u)||k_\lambda(u)| \, du \, dt \\
&= \iint_{\mathbb{R}^2} |f(t) - f(t-u)||k_\lambda(u)| \, dt \, du \quad \text{[Tonelli]} \\
&= \int_{\mathbb{R}} |k_\lambda(u)| \int_{\mathbb{R}} |f(t) - f(t-u)| \, dt \, du \\
&= \int_{\mathbb{R}} |k_\lambda(u)| \|f - f_u\|_1 \, du
\end{aligned}$$

Using Theorem 2.7 we know that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|u| < \delta \Longrightarrow \|f - f_u\|_1 < \varepsilon$$

Also, using the properties of an approximate identity we have

$$K = \sup_{\lambda > 0} \|k_\lambda\|_1 < \infty$$

and there exists some $\lambda_0 > 0$ such that

$$\lambda < \lambda_0 \Longrightarrow \int_{|u| \ge \delta} |k_\lambda(u)| \, du < \varepsilon$$

So now assume that $\lambda < \lambda_0$ and continue the calculation we started above:

$$\begin{aligned}
\|f - f * k_\lambda\|_1 &= \int_{\mathbb{R}} |k_\lambda(u)| \|f - f_u\|_1 \, du \\
&= \int_{|u|<\delta} |k_\lambda(u)| \|f - f_u\|_1 \, du + \int_{|u|\geq\delta} |k_\lambda(u)| \|f - f_u\|_1 \, du \\
&\leq \int_{|u|<\delta} |k_\lambda(u)| \varepsilon \, du + \int_{|u|\geq\delta} |k_\lambda(u)| (\|f\|_1 + \|f_u\|_1) \, du \\
&\leq \varepsilon \int_{\mathbb{R}} |k_\lambda(u)| \, du + 2\|f\|_1 \int_{|u|\geq\delta} |k_\lambda(u)| \, du \\
&\leq \varepsilon K + 2\|f\|_1 \varepsilon
\end{aligned}$$

Taking $\varepsilon \to 0$ and $\lambda_0 \to 0$ completes the proof. □

Exercise 4. Prove that the assumptions of Theorem 2.2 (Fourier inversion) imply that f must be continuous and bounded.

Remark 2.8. Exercise 4 shows that our Fourier inversion theorem only holds for continuous functions. However, many signals that we encounter will have discontinuities. Thus we will need to extend the theory to discontinuous functions. This will be done by extending the Fourier transform to $\mathbf{L}^2(\mathbb{R})$ (more on this later).

Recall in Section 2.1 that for a linear shift invariant operator L with impulse response h, we wanted to write Lf in terms of the eigenvalues $\widehat{h}(\omega)$ of L by also being able to compute $\widehat{f}(\omega)$. The previous theorem gives us part of the solution; the other part is given by the *convolution theorem*, which is stated next.

Theorem 2.9 (Convolution thoerem)**.** *Let $f, g \in \mathbf{L}^1(\mathbb{R})$. Then the function $h = f * g \in \mathbf{L}^1(\mathbb{R})$ and*

$$\widehat{h}(\omega) = \widehat{g}(\omega)\widehat{f}(\omega)$$

Proof. See p. 37 of *A Wavelet Tour of Signal Processing.* □

Recall now that every bounded, linear shift invariant operator L can be written as $Lf = h * f$, where $h = L\delta$. Thus using the Fourier inversion theorem and the convolution theorem we have:

$$Lf(t) = h * f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{h * f}(\omega) e^{i\omega t} \, d\omega = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{h}(\omega)\widehat{f}(\omega) e^{i\omega t} \, d\omega$$

Thus at last we see that the sinusoids $e_\omega(t) = e^{i\omega t}$ diagonalize L, with eigenvalues $\widehat{h}(\omega)/2\pi$. To see this recall from before we wrote for a real valued symmetric matrix $\mathbf{A}$ with eigenvectors $\{v_k\}_k$ and eigenvalues $\{\lambda_k\}_k$,

$$\mathbf{A}x = \sum_{k=1}^{n} \langle x, v_k \rangle \mathbf{A} v_k = \sum_{k=1}^{n} \lambda_k \langle x, v_k \rangle v_k \, .$$

Property	Function	Fourier Transform
	$f(t)$	$\hat{f}(\omega)$
Inverse	$\hat{f}(t)$	$2\pi f(-\omega)$
Convolution	$f_1 \star f_2(t)$	$\hat{f}_1(\omega)\hat{f}_2(\omega)$
Multiplication	$f_1(t)\, f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$
Translation	$f(t-u)$	$e^{-iu\omega}\hat{f}(\omega)$
Modulation	$e^{i\xi t} f(t)$	$\hat{f}(\omega-\xi)$
Scaling	$f(t/s)$	$\|s\|\hat{f}(s\,\omega)$
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p \hat{f}(\omega)$
Frequency derivatives	$(-it)^p f(t)$	$\hat{f}^{(p)}(\omega)$
Complex conjugate	$f^*(t)$	$\hat{f}^*(-\omega)$
Hermitian symmetry	$f(t) \in \mathbb{R}$	$\hat{f}(-\omega) = \hat{f}^*(\omega)$

Figure 1: Summary of basic properties of the Fourier transform. Taken from Table 2.1 of *A Wavelet Tour of Signal Processing*.

The correspondence is $\lambda_k \leftrightarrow \widehat{h}(\omega)/2\pi$, $\langle x, v_k \rangle \leftrightarrow \widehat{f}(\omega)$, and $v_k \leftrightarrow e^{i\omega t}$.

The Fourier transform has several important properties that are listed in Figure 1.

Exercise 5. Verify all of the properties in Figure 1. No need to turn this one in, but it is important to do these verifications.

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.

[3] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.