Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 03: The Fourier transform on $\mathbf{L}^2(\mathbb{R})$

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Remark 2.10. The Dirac $\delta(t)$ is not a function, and hence is not in $\mathbf{L}^1(\mathbb{R})$; it is a distribution, which we will not discuss in this course. The Dirac distribution has the property of being an identity under convolution, meaning that $f * \delta(u) = f(u)$ if $f \in \mathbf{L}^1(\mathbb{R})$ and if f is continuous. There is no $\mathbf{L}^1(\mathbb{R})$ function with this property, so the question is how to define the Dirac. The notion of an approximate identity $\{k_\lambda\}_{\lambda>0}$, defined above, can be used to define it. Indeed, we define $\delta(t) = \lim_{\lambda\to 0} k_\lambda(t)$, where we understand that the limit means *weak convergence*. By weak convergence, we mean that for any continuous function ϕ,

$$\lim_{\lambda\to 0} \int_{\mathbb{R}} \phi(t) k_\lambda(t)\, dt = \phi(0) =: \int_{\mathbb{R}} \phi(t)\delta(t)\, dt$$

We can also define a translated Dirac $\delta_\tau(t) = \delta(t-\tau)$, which is defined as the weak limit of a translated approximate identity. This means that

$$\phi * \delta(u) = \int_{\mathbb{R}} \phi(t)\delta(u-t)\, dt = \int_{\mathbb{R}} \phi(t)\delta(t-u)\, dt = \phi(u) \tag{5}$$

Note that these properties, in particular (5), follow from defining $\delta(t) = \lim_{\lambda\to 0} k_\lambda(t)$ in the weak sense, Theorem 2.6, and the fact that ϕ is continous.

Using this formalism, we define the Fourier transform of $\delta(t)$, $\widehat{\delta}(\omega)$, as:

$$\widehat{\delta}(\omega) = \int_{\mathbb{R}} \delta(t) e^{-i\omega t}\, dt = 1$$

A translated Dirac $\delta_\tau(t) = \delta(t-\tau)$ has Fourier transform calculated by evaluating $e^{-i\omega t}$ at $t = \tau$,

$$\widehat{\delta_\tau}(\omega) = \int_{\mathbb{R}} \delta(t-\tau) e^{-i\omega t}\, dt = e^{-i\omega\tau}$$

The *Dirac comb* is a sum of translated Diracs:

$$c(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \tag{6}$$

It is used to obtain a discrete sampling of an analogue signal, as we shall see later. Its Fourier transform is:

$$\widehat{c}(\omega) = \sum_{n=-\infty}^{+\infty} e^{-inT\omega}$$

Remarkably, $\widehat{c}(\omega)$ is also a Dirac comb, as the next theorem shows.

Theorem 2.11 (Poisson Formula)**.** *In the sense of distribution equalities,*

$$\sum_{n=-\infty}^{+\infty} e^{-inT\omega} = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

In other words, for every $\widehat{\phi} \in \mathbf{C}_0^\infty(\mathbb{R})$*, that is for every compactly supported infinitely differentiable function, one has*

$$\int_{\mathbb{R}} \widehat{\phi}(\omega) \left[\sum_{n=-\infty}^{+\infty} e^{-inT\omega}\right] d\omega = \int_{\mathbb{R}} \widehat{\phi}(\omega) \left[\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)\right] d\omega$$

Proof. See p. 41–42 of *A Wavelet Tour of Signal Processing.* □

Consider now the function

$$f(t) = \mathbf{1}_{[-1,1]}(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We can compute the Fourier transform of this function:

$$\widehat{f}(\omega) = \int_{-1}^{1} e^{-i\omega t}\, dt = \frac{2\sin(\omega)}{\omega}$$

One can verify that this function is not integrable; we would expect this from Exercise 4 because $f(t)$ is not continuous. However, $\widehat{f}(\omega)$ is square integrable; that is $\widehat{f} \in \mathbf{L}^2(\mathbb{R})$. This motivates extending the Fourier transform to functions $f \in \mathbf{L}^2(\mathbb{R})$. Recall that $\mathbf{L}^2(\mathbb{R})$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t) g^*(t)\, dt.$$

We first have the following fundamental results:

Theorem 2.12 (Parseval)**.** *Let* $f, g \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$*. Then:*

$$\langle f, g\rangle = \frac{1}{2\pi} \langle \widehat{f}, \widehat{g}\rangle$$

Proof. See p. 39 of *A Wavelet Tour of Signal Processing.* □

Corollary 2.13 (Plancheral)**.** *Let* $f \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$*. Then:*

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}} \|\widehat{f}\|_2$$

Note that in the previous theorems, the inner product and norm are computable because we assume $f, g \in \mathbf{L}^2(\mathbb{R})$, but the Fourier transform is only well defined because we assume $f, g \in \mathbf{L}^1(\mathbb{R})$ as well. We would like to remedy this be extending the Fourier transform to all functions $f \in \mathbf{L}^2(\mathbb{R})$, even those for which $f \notin \mathbf{L}^1(\mathbb{R})$. We do this with a *density argument*, which will define the Fourier transform of a function $f \in \mathbf{L}^2(\mathbb{R})$ as the limit of Fourier transforms of functions in $\mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$. A very useful inequality from real analysis, which we will need here, is *Hölder's inequality*:

$$\forall\, f \in \mathbf{L}^p(\mathbb{R}),\ g \in \mathbf{L}^q(\mathbb{R}),\ p, q \in [1, \infty],\ \frac{1}{p} + \frac{1}{q} = 1, \quad \|fg\|_1 \leq \|f\|_p \|g\|_q$$

Now to the density argument. The first thing to note is that $\mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ is dense in $\mathbf{L}^2(\mathbb{R})$. This means that given an $f \in \mathbf{L}^2(\mathbb{R})$, we can find a family $\{f_n\}_{n \geq 1}$ of functions in $\mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ that converges to f,

$$\lim_{n \to \infty} \|f - f_n\|_2 = 0$$

In fact it is easy to find a such a family. Define:

$$f_n(t) = f(t)\mathbf{1}_{[-n,n]}(t)$$

We have that $f_n \in \mathbf{L}^2(\mathbb{R})$ for all $n \geq 1$ since $|f_n(t)| \leq |f(t)|$ for all $t \in \mathbb{R}$. Furthermore, $f_n \in \mathbf{L}^1(\mathbb{R})$ since by Hölder's inequality we have:

$$\begin{aligned}
\|f_n\|_1 = \int_{\mathbb{R}} |f_n(t)|\, dt &= \int_{\mathbb{R}} |f(t)\mathbf{1}_{[-n,n]}(t)|\, dt \\
&\leq \left(\int_{\mathbb{R}} |f(t)|^2\, dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\mathbf{1}_{[-n,n]}(t)|^2 \right)^{\frac{1}{2}} \\
&= \|f\|_2 \left(\int_{-n}^{n} 1\, dt \right)^{\frac{1}{2}} \\
&= \sqrt{2n} \|f\|_2
\end{aligned}$$

We also have that

$$\|f - f_n\|_2 = \left(\int_{|t|>n} |f(t)|^2 \right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty$$

Now, since $f_n \to f$, the family $\{f_n\}_{n \geq 1}$ is also a *Cauchy sequence*, meaning that for all $\varepsilon > 0$ there exists an N such that if $n, m > N$, then $\|f_n - f_m\|_2 \leq \varepsilon$. Furthermore, since $f_n \in \mathbf{L}^1(\mathbb{R})$, its Fourier transform $\widehat{f}_n$ is well defined. The Plancheral formula (Corollary 2.13) then yields:

$$\|\widehat{f}_n - \widehat{f}_m\|_2 = \sqrt{2\pi} \|f_n - f_m\|_2$$

Thus since $\{f_n\}_{n \geq 1}$ is a Cauchy sequence, we see that $\{\widehat{f}_n\}_{n \geq 1}$ is a Cauchy sequence as well. Since $\mathbf{L}^2(\mathbb{R})$ is a Hilbert space, it is complete, which means that every Cauchy sequence converges to an element of $\mathbf{L}^2(\mathbb{R})$. Thus there exists an $F \in \mathbf{L}^2(\mathbb{R})$ such that

$$\lim_{n \to \infty} \|F - \widehat{f}_n\|_2 = 0$$

We define the Fourier transform of $f \in \mathbf{L}^2(\mathbb{R})$ as F, and from now on write $\widehat{f} = F$. Note that when $f \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ this definition of the Fourier transform (the $\mathbf{L}^2$ definition) coincides with the definition given in (1) (the $\mathbf{L}^1$ definition).

One can show the extension of the Fourier transform to $\mathbf{L}^2(\mathbb{R})$ satisfies the convolution theorem (Theorem 2.9), the Parseval formula (Theorem 2.12), the Plancheral formula (Corollary 2.13), and all properties in Figure 1. In particular, the Plancheral formula implies the following. Let $\mathcal{F}(f) = \widehat{f}$, so that $\mathcal{F}$ is the operator that maps a function f to its Fourier transform $\widehat{f}$. We see from the Plancheral formula and the extension of the Fourier transform to $\mathbf{L}^2(\mathbb{R})$ that $\mathcal{F} : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R})$, and furthermore that this linear operator is an isometry up to a factory $1/\sqrt{2\pi}$. The operator $\mathcal{F} : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R})$ is bijective, and thus is invertible; we therefore have Fourier inversion for $\mathbf{L}^2(\mathbb{R})$ functions as well.

Remark 2.14. To summarize the Fourier transform can be defined on $\mathbf{L}^1(\mathbb{R})$ in which case we have

$$\mathcal{F} : \mathbf{L}^1(\mathbb{R}) \to \mathbf{L}^\infty(\mathbb{R})$$

with $\|\widehat{f}\|_\infty \leq \|f\|_1$, or on $\mathbf{L}^2(\mathbb{R})$ where we have:

$$\mathcal{F} : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R})$$

with $\|f\|_2 = (1/\sqrt{2\pi})\|\widehat{f}\|_2$. It follows then from the Riesz-Thorin Theorem that the Fourier transform can be extended to $\mathbf{L}^p(\mathbb{R})$ for any $1 \leq p \leq 2$, where we have

$$\mathcal{F} : \mathbf{L}^p(\mathbb{R}) \to \mathbf{L}^q(\mathbb{R}), \quad \frac{1}{p} + \frac{1}{q} = 1, \;\; 1 \leq p \leq 2$$

and that

$$\|\widehat{f}\|_q \leq \left(\frac{1}{2\pi}\right)^{\frac{1}{p}} \|f\|_p \tag{7}$$

Equation (7) is called the *Hausdorff–Young Inequality.* Note that in general one only obtains equality for $p = q = 2$, and indeed $\mathcal{F}$ is not an isometry otherwise (up to the constant factor) and is not invertible. Indeed, we saw this for $\mathbf{L}^1(\mathbb{R})$, where in order to get Fourier inversion we had to assume that $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$ as well.

Exercise 6. Read Section 2.2 of *A Wavelet Tour of Signal Processing.*

2.3 Regularity and Decay

Section 2.3.1 of A Wavelet Tour of Signal Processing [1].

The global regularity of f depends on the decay of $|\widehat{f}(\omega)|$ as $\omega \to \infty$. In particular, the smoother the function, the faster the decay of $|\widehat{f}(\omega)|$. The intuition is that smooth functions vary slowly, and thus can be well represented by low frequency modes $e^{i\omega t}$, i.e., those with small values of $|\omega|$. On the other hand, if f is irregular, then it must have sharp transitions which require fast oscillations to capture. We make these intuitions precise with the following two results. First define $\mathbf{C}^n(\mathbb{R})$ as the space of functions with n continuous derivatives; $\mathbf{C}^0(\mathbb{R})$ is the space of continuous functions.

Theorem 2.15. *Let $1 \le p \le 2$ and $f \in \mathbf{L}^p(\mathbb{R})$. If there exits a constant C and $\epsilon > 0$ such that*

$$|\widehat{f}(\omega)| \le \frac{C}{1 + |\omega|^{n+1+\epsilon}}$$

for some $n \in \mathbb{N}$, then $f \in \mathbf{C}^n(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$.

Proof. We know from Exercise 4 that if $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$, then f is continuous and bounded. Notice for $n = 0$ we have:

$$\|\widehat{f}\|_1 = \int_{\mathbb{R}} |\widehat{f}(\omega)|\, d\omega \le \int_{\mathbb{R}} \frac{C}{1 + |\omega|^{1+\epsilon}}\, d\omega < \infty$$

So indeed $f \in \mathbf{C}(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$. Now consider $n \in \mathbb{N}$ and $k \le n$; define the function $F_k(\omega) = (i\omega)^k \widehat{f}(\omega)$. We see that:

$$\|F_k\|_1 \le \int_{\mathbb{R}} \frac{C|\omega|^k}{1 + |\omega|^{n+1+\epsilon}}\, d\omega < \infty$$

It thus follows that $\mathcal{F}^{-1}(F_k)$ (i.e., the inverse Fourier transform of F_k) is continuous and bounded. But from Figure 1 we know that $\mathcal{F}^{-1}(F_k) = f^{(k)}(t)$, and so the proof is completed. □

Note in particular that if $\widehat{f}$ has compact support, then $f \in \mathbf{C}^\infty(\mathbb{R})$. In the other direction we have:

Theorem 2.16. *Let $f \in \mathbf{C}^n(\mathbb{R})$ with $f^{(n)} \in \mathbf{L}^1(\mathbb{R})$. Then:*

$$|\widehat{f}(\omega)| \le \frac{C}{|\omega|^n}$$

for some constant C.

Exercise 7. Prove Theorem 2.16.

Remark 2.17. Notice there is a gap between the two theorems relating regularity and decay. This cannot be avoided. Furthermore, we notice that the decay of $|\widehat{f}(\omega)|$ depends upon the *worst* singular behavior of f. Indeed as the function $f(t) = \mathbf{1}_{[-1,1]}(t)$ illustrates, the function is discontinuous and thus its Fourier decay is limited by Theorem 2.15. However, f has only two singular points. It is often much more desirable to characterize the local regularity of a function. However, the Fourier transform cannot do this since the sinusoids $e^{i\omega t}$ are global functions on $\mathbb{R}$. In order to remedy both of these points, we will need to introduce localized waveforms. We will see later that wavelets do the job.

Exercise 8. Show that the Fourier transform of

$$f(t) = e^{-(a-ib)t^2}, \quad a > 0$$

is

$$\widehat{f}(\omega) = \sqrt{\frac{\pi}{a-ib}} \exp\left(-\frac{a+ib}{4(a^2+b^2)}\omega^2\right)$$

Exercise 9 (Riemann-Lebesgue Lemma)**.** Prove that if $f \in \mathbf{L}^1(\mathbb{R})$, then $\lim_{|\omega|\to\infty} \widehat{f}(\omega) = 0$. Hint: Start with $f \in \mathbf{C}^1(\mathbb{R})$ that have compact support, and use a density argument. This approach uses the standard fact from real analysis that compactly supported $\mathbf{C}^\infty(\mathbb{R})$ functions are dense in $\mathbf{L}^1(\mathbb{R})$. However, if you have not seen this before, it is unsatisfying to use it here to prove the exercise. In this case, consider instead the Gaussian function:

$$g(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

Define dilations of g as:

$$g_\sigma(u) = \sigma^{-1} g(\sigma^{-1} u), \quad \sigma > 0$$

Prove that $\{g_\sigma\}_{\sigma>0}$ forms an approximate identity. The functions $\{f * g_\sigma\}_{\sigma>0}$ are not compactly supported, but they can be used to prove the result. Figure out how and provide the proof.

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.

[3] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.