Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 05: Fourier Series

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3.2 Fourier Series

Section 3.2.2 of A Wavelet Tour of Signal Processing.

Let the sampling rate be $s = 1$ now, which gives samples $\{f(n)\}_{n\in\mathbb{Z}}$ of a signal $f(t)$. Previously we defined

$$f_d(t) = \sum_{n\in\mathbb{Z}} f(n)\delta(t-n)$$

and observed that

$$\widehat{f_d}(\omega) = \sum_{n\in\mathbb{Z}} f(n)e^{-in\omega}$$

This is a *Fourier series*. Clearly $\widehat{f_d}(\omega)$ is 2π periodic, and thus it is uniquely determined by its restriction to $[-\pi,\pi]$. This motivates defining Fourier series on $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^2$, which will allow us to represent $f_d(t)$ as a sequence $a = (a[n])_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^p$ with $a[n] = f(n)$. Define

$$\boldsymbol{\ell}^p = \left\{ a = (a[n])_{n\in\mathbb{Z}} : a[n] \in \mathbb{C} \text{ and } \sum_{n\in\mathbb{Z}} |a[n]|^p < \infty \right\}, \quad 0 < p < \infty$$

and

$$\boldsymbol{\ell}^\infty = \left\{ a = (a[n])_{n\in\mathbb{Z}} : a[n] \in \mathbb{C} \text{ and } \sup_{n\in\mathbb{Z}} |a[n]| < \infty \right\}$$

Define the Fourier transform of $a \in \boldsymbol{\ell}^p$ as:

$$\mathcal{F}(a)(\omega) = \widehat{a}(\omega) = \sum_{n\in\mathbb{Z}} a[n]e^{-in\omega}, \quad \omega \in [-\pi,\pi]$$

The Fourier transform of $a \in \boldsymbol{\ell}^1$ is a Fourier series; it is analogous to the Fourier transform of $f \in \mathbf{L}^1(\mathbb{R})$. The spaces $\mathbf{L}^p[-\pi,\pi]$ are defined the same as $\mathbf{L}^p(\mathbb{R})$, except that the domain $\mathbb{R}$ is replaced with $[-\pi,\pi]$, and we normalize the norm so that for $A \in \mathbf{L}^p[-\pi,\pi]$ we have

$$\|A\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |A(\omega)|^p \, d\omega \right)^{1/p}$$

For $\mathbf{L}^2[-\pi,\pi]$ we have the inner product defined as:

$$\langle A, B \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega)B^*(\omega)\, d\omega$$

It is easy to see that $\mathcal{F} : \boldsymbol{\ell}^1 \to \mathbf{L}^\infty[-\pi,\pi]$ and with a little more work (see Theorem 3.3 below) that $\mathcal{F} : \boldsymbol{\ell}^2 \to \mathbf{L}^2[-\pi,\pi]$, which mirrors our results for the Fourier transform defined on $\mathbf{L}^1(\mathbb{R})$ and $\mathbf{L}^2(\mathbb{R})$. Further developing the parallel story, Theorem 3.3 below shows that the family of functions $\{e_n\}_{n\in\mathbb{Z}}$ with

$$e_n(\omega) = e^{-in\omega}$$

is an orthonormal basis for $\mathbf{L}^2[-\pi,\pi]$. It follows that $\mathcal{F} : \boldsymbol{\ell}^2 \to \mathbf{L}^2[-\pi,\pi]$ is a bijection, and hence invertible.

Theorem 3.3. *The family of functions $\{e_n\}_{n\in\mathbb{Z}}$ is a an orthonormal basis for $\mathbf{L}^2[-\pi,\pi]$.*

Proof. The proof that $\{e_n\}_{n\in\mathbb{Z}}$ are orthonormal is by direct calculation of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\omega}e^{im\omega}\,d\omega = \begin{cases} 1 & n=m \\ 0 & n\neq m \end{cases}$$

Now we must show that linear expansions of $\{e_n\}_{n\in\mathbb{Z}}$ are dense in $\mathbf{L}^2[-\pi,\pi]$. This means we need to show the following: Let $A \in \mathbf{L}^2[-\pi,\pi]$, $N > 0$ and define the partial Fourier series of A as:

$$S_N(\omega) = \sum_{n=-N}^{N} \langle A, e_n\rangle e^{-in\omega}, \quad \langle A, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)e^{in\omega}\,d\omega$$

We need to show that for each $\varepsilon > 0$, there exists $N > 0$ such that

$$\|A - S_N\|_2 < \varepsilon$$

which will mean that $\lim_{N\to\infty} S_N = A$ (in the $\mathbf{L}^2$ sense), which we can write as

$$A(\omega) = \sum_{n\in\mathbb{Z}} \langle A, e_n\rangle e^{-in\omega}$$

To prove this we will use some facts about periodic functions; the proofs of these results can be found in [2]. To start, define a trigonometric polynomial $P(\omega)$ as any function

$$P(\omega) = \sum_{n\in\mathbb{Z}} a_n e^{-in\omega}$$

with only a finite number of coefficients a_n being non-zero. The degree of P is defined as the largest value $|n|$ such that $a_n \neq 0$. One fact we will need is that any function $\phi \in \mathbf{C}[-\pi,\pi]$ with $\phi(-\pi) = \phi(\pi)$ can be uniformly approximated by trigonometric polynomials. That is, for each such ϕ and each $\varepsilon > 0$ there exists P such that

$$|\phi(\omega) - P(\omega)| \leq \varepsilon, \quad \forall -\pi \leq \omega \leq \pi$$

A second fact we will need is that for $A \in \mathbf{L}^2[-\pi,\pi]$ and $\varepsilon > 0$, we can find a function $\phi \in \mathbf{C}[-\pi,\pi]$ with $\phi(-\pi) = \phi(\pi)$ such that

$$\|\phi\|_\infty \le \|A\|_\infty$$

and

$$\|A - \phi\|_2 \le \varepsilon^2$$

We also remark that $A \in \mathbf{L}^2[-\pi,\pi]$ implies that $A \in \mathbf{L}^p[-\pi,\pi]$ for any since the length of $[-\pi,\pi]$ is finite.

Now for the remainder of the proof. Since the family $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal, we must have

$$A - S_N \perp e_n, \quad \forall\, |n| \le N$$

from which it follows that $A - S_N \perp P_N$, where P_N is any trigonometric polynomial of degree N. Taking $P_N = S_N$, this in turn gives:

$$\|A\|_2^2 = \|A - S_N + S_N\|_2^2 = \|A - S_N\|^2 + \|S_N\|^2$$

We also note that for any trigonometric polynomial P_N, we have

$$\|A - S_N\|_2 \le \|A - P_N\|_2 \tag{11}$$

with equality only when $P_N = S_N$. Indeed:

$$A - P_N = A - S_N + \underbrace{(S_N - P_N)}_{\widetilde{P}_N}$$

which implies that

$$\|A - P_N\|_2^2 = \|A - S_N\|_2^2 + \|\widetilde{P}_N\|_2^2$$

from which the inequality (11) follows.

We now complete the proof. First consider a $\phi \in \mathbf{C}[-\pi,\pi]$ with $\phi(-\pi) = \phi(\pi)$. Given $\varepsilon > 0$, we can find a trigonometric polynomial P_M with degree M such that

$$|\phi(\omega) - P_M(\omega)| < \varepsilon, \quad \forall -\pi \le \omega \le \pi$$

Therefore:

$$\begin{aligned} \|\phi - P_M\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi(\omega) - P_M(\omega)|^2\, d\omega \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \varepsilon^2\, d\omega \le \varepsilon^2 \end{aligned}$$

Thus we have $\|\phi - P_M\| \le \varepsilon$. But since partial Fourier series are the best approximation (11), we then conclude that

$$\|\phi - S_N(\phi)\|_2 \le \|\phi - P_M\|_2 \le \varepsilon, \quad \forall\, N \ge M$$

Now let us return to the case of general $A \in \mathbf{L}^2[-\pi, \pi]$. For $\varepsilon > 0$, approximate A with a $\phi \in \mathbf{C}[-\pi, \pi]$, $\phi(-\pi) = \phi(\pi)$, such that $\|A - \phi\|_2 \leq \varepsilon$. Approximate ϕ with a trigonometric polynomial P_M, as before, to obtain:

$$\|A - P_M\|_2 \leq \|A - \phi\|_2 + \|\phi - P_M\|_2 \leq 2\varepsilon$$

Now again use the best approximation inequality (11) to conclude that:

$$\|A - S_N\|_2 \leq 2\varepsilon, \quad \forall N \geq M$$

□

Theorem 3.3 proves that any periodic function $A \in \mathbf{L}^2[-\pi, \pi]$ can be written as

$$A(\omega) = \sum_{n \in \mathbb{Z}} c_n e^{-in\omega} \tag{12}$$

with

$$c_n = \langle A, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\omega) e^{in\omega} \, d\omega \tag{13}$$

In particular, if we start with $a \in \boldsymbol{\ell}^2$ and compute its Fourier series:

$$\widehat{a}(\omega) = \sum_{n \in \mathbb{Z}} a[n] e^{-in\omega}$$

then we must have

$$a[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}(\omega) e^{in\omega} \, d\omega$$

which is a type of Fourier inversion formula for Fourier series. Additionally, if $a[n] = f(n)$ for some signal f, then from the beginning of this section we see that $\widehat{f_d}(\omega) = \widehat{a}(\omega)$ and

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f_d}(\omega) e^{in\omega} \, d\omega$$

which gives Fourier inversion for the samples $f(n)$ from $\widehat{f_d}(\omega)$. We have a similar distributional version, which requires defining one more distributional Fourier transform. Recall that for $\delta_\tau(t) = \delta(t - \tau)$, we defined the Fourier transform as $\widehat{\delta_\tau}(\omega) = e^{-i\omega\tau}$. Given this, if we set $e_\xi(t) = e^{i\xi t}$, it makes sense to define its Fourier transform as $\widehat{e_\xi}(\omega) = 2\pi\delta(\omega - \xi)$. Indeed, e_ξ is a perfect harmonic vibrating at frequency ξ. Using this fact, if we compute the inverse Fourier transform of $\widehat{f_d}(\omega)$ in the distributional sense, it is clear we will get back $f_d(t)$. More generally, if we start $A \in \mathbf{L}^2[-\pi, \pi]$ and write it as in (12), and compute the distributional inverse Fourier transform of A, we will get

$$\mathcal{F}^{-1}(A)(t) = \sum_{n \in \mathbb{Z}} \langle A, e_n \rangle \delta(t - n)$$

Of course this is equivalent to computing the Fourier series inversion of A and getting a sequence $(\langle A, e_n\rangle)_{n\in\mathbb{Z}} \in \boldsymbol{\ell}^2(\mathbb{R})$, but it is sometimes convenient to use one over the other.

We also have the following version of the Plancheral formula:

$$\|a\|^2 = \sum_{n\in\mathbb{Z}} |a[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{a}(\omega)|^2\, d\omega = \|\widehat{a}\|^2$$

Define the convolution of $a, b \in \boldsymbol{\ell}^1$ as:

$$a * b[n] = \sum_{m\in\mathbb{Z}} a[m]b[n-m]$$

We have a convolution theorem for $\boldsymbol{\ell}^1$ sequences as well:

Theorem 3.4. *Let $a, b \in \boldsymbol{\ell}^1$. Then $a * b \in \boldsymbol{\ell}^1$ and*

$$\widehat{a * b}(\omega) = \widehat{a}(\omega)\widehat{b}(\omega)$$

Remark 3.5. (see also [4]) In this section we started with a sequence $a \in \boldsymbol{\ell}^2$ and defined its Fourier transform as the Fourier series with coefficients $a[n]$. We saw the resulting Fourier series defines a periodic function $A(\omega)$ on $\mathbf{L}^2[-\pi, \pi]$. This was motivated by discrete sampling of a function $f \in \mathbf{L}^2(\mathbb{R})$. In many (pure) harmonic analysis texts, though, one starts with a periodic function A on $[-\pi, \pi]$ and computes its Fourier coefficients as the inner products of A with e_n, as in (13). In Theorem 3.3 we proved that for any $A \in \mathbf{L}^2[-\pi, \pi]$, the partial Fourier series

$$S_N(A)(\omega) = \sum_{|n|\leq N} \langle A, e_n\rangle e^{-in\omega}$$

converges to A in $\mathbf{L}^2$ norm as $N \to \infty$, i.e.,

$$\lim_{N\to\infty} \|S_N(A) - A\|_2 = 0$$

In fact, the same is true for any $A \in \mathbf{L}^p[-\pi, \pi]$, for $1 < p < \infty$, that is:

$$\lim_{N\to\infty} \|S_N(A) - A\|_p = 0$$

Note this is not true for $p = 1$ or $p = \infty$! The Plancheral formula above shows that

$$\|A\|_2^2 = \sum_{n\in\mathbb{Z}} |\langle A, e_n\rangle|^2$$

that is, if we know the amplitudes of the Fourier coefficients of A, then we can deduce its $\mathbf{L}^2$ norm. It turns out the same is not true for $2 < p < \infty$; knowing the amplitudes is not enough, the phases play a critical role. Let us consider a sequence $a \in \boldsymbol{\ell}^2$. Let ε_n, $n \in \mathbb{Z}$, be a sequence of independent Bernoulli random variables, meaning they take value +1 with probability 1/2 and value −1 with probability 1/2. Then the random Fourier series

$$\sum_{|n|\leq N} \varepsilon_n a[n] e^{-in\omega}$$

converges to a function that, for $2 < p < \infty$, belongs to each $\mathbf{L}^p[-\pi, \pi]$, almost surely (i.e., for almost all choices of the sequence ε_n). But it does not converge for all random sequences! If we want to estimate the $\mathbf{L}^p$ norm of A, breaking it down into its individual Fourier coefficients goes too far. The right approach is to break the frequencies down into dyadic packets. These are defined as:

$$\Delta_j A(\omega) = \sum_{2^j \leq |n| < 2^{j+1}} \langle A, e_n \rangle e^{-in\omega}, \quad j \in \mathbb{N}$$

Then using Littlewood-Paley theory, we have that the norm $\|A\|_p$ is equivalent to

$$|\langle A, e_0 \rangle| + \left\| \left(\sum_{j=0}^{\infty} |\Delta_j A|^2 \right)^{1/2} \right\|_p$$

This means, in particular, if $\sum_{j=0}^{\infty} \Delta_j A \in \mathbf{L}^p[-\pi, \pi]$, then so does $\sum_{j=0}^{\infty} \varepsilon_j \Delta_j A$ for all choices of $\varepsilon_j = \pm 1$. This type of idea, breaking the frequencies down into dyadic packets, is at the heart of wavelet analysis, which we will get to later. We will also rely on this type of Littlewood-Paley analysis when we want to analyze $\mathbf{C}^\alpha$ functions with wavelets.

Exercise 14. Read Section 3.2 of *A Wavelet Tour of Signal Processing.*

Exercise 15. A rectifier computes $g(t) = |f(t)|$ for recovering the envelope of modulated signals.

(a) Show that if $f(t) = h(t)\sin(\omega_0 t)$ with $h \in \mathbf{L}^1(\mathbb{R})$, $h(t) \geq 0$ and $\omega_0 > 0$, then $g(t) = |f(t)|$ satisfies

$$\widehat{g}(\omega) = \frac{2}{\pi} \sum_{n=-\infty}^{+\infty} \frac{\widehat{h}(\omega - 2n\omega_0)}{4n^2 - 1}$$

Hint: Let $A(t)$ be a 2π periodic function. By Theorem 3.3 we can write

$$A(t) = \sum_{n\in\mathbb{Z}} c_n e^{int}$$

with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(t) e^{-int}\, dt$$

Compute now the Fourier transform of A in the distributional sense. Do this for $A(t) = |\sin(t)|$ and apply the convolution theorem to $g(t)$ (along with the rescaling property of the Fourier transform).

(b) Suppose that $\widehat{h}(\omega) = 0$ for $|\omega| \geq \omega_0$. Find ϕ such that $h(t) = \phi * g(t)$.

Exercise 16. An *interpolation function* $f(t)$ satisfies $f(n) = \delta(n)$ for any $n \in \mathbb{Z}$.

(a) Prove that

$$\sum_{n\in\mathbb{Z}} \widehat{f}(\omega + 2n\pi) = 1 \quad \Longleftrightarrow \quad f \text{ is an interpolation function}$$

(b) Suppose that

$$f(t) = \sum_{n\in\mathbb{Z}} a[n]\theta(t-n), \quad a \in \boldsymbol{\ell}^1, \;\; \theta \in \mathbf{L}^2(\mathbb{R})$$

Find $\widehat{a}(\omega)$ as a function of $\widehat{\theta}(\omega)$ so that $f(t)$ is an interpolation function. Relate $\widehat{f}(\omega)$ to $\widehat{\theta}(\omega)$, and give a sufficient condition on $\widehat{\theta}$ to guarantee that $f \in \mathbf{L}^2(\mathbb{R})$.

Exercise 17. Let $g \in \boldsymbol{\ell}^1$ and set $h[n] = (-1)^n g[n]$. Relate $\widehat{h}(\omega)$ to $\widehat{g}(\omega)$. If g is a low pass filter (meaning that $\widehat{g}(\omega)$ is concentrated around 0), then what kind of filter is h? (i.e., where is its support concentrated?)

Exercise 18. Let $b \in \boldsymbol{\ell}^1$. A *decimation* of b computes a signal $a \in \boldsymbol{\ell}^1$ with $a[n] = b[Mn]$ for $M > 1$ $(M \in \mathbb{Z})$.

(a) Show that

$$\widehat{a}(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} \widehat{b}(M^{-1}(\omega - 2k\pi))$$

(b) Give a sufficient condition on $\widehat{b}(\omega)$ to recover b from a and give the interpolation formula that recovers $b[n]$ from a.

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.

[3] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.

[4] Yves Meyer. *Wavelets and Operators*, volume 1. Cambridge University Press, 1993.