Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 07: Time-Frequency Analysis and Windowed Fourier

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4 Time Meets Frequency

Chapter 4 of A Wavelet Tour of Signal Processing [1].

4.1 Time Frequency Atoms

Section 4.1 of A Wavelet Tour of Signal Processing [1].

A linear *time frequency transform* correlates the signal $f(t)$ with a dictionary of waveforms that are concentrated in time and frequency; these waveforms are called time frequency atoms. Denote a general dictionary of time frequency atoms by:

$$\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}, \quad \phi_\gamma \in \mathbf{L}^2(\mathbb{R}), \ \ \|\phi_\gamma\|_2 = 1$$

where Γ is a (multi)-index set. The time frequency transform of $f \in \mathbf{L}^2(\mathbb{R})$ in the dictionary $\mathcal{D}$ computes

$$\Phi f(\gamma) = \langle f, \phi_\gamma \rangle = \int_{\mathbb{R}} f(t)\phi_\gamma^*(t)\, dt$$

Recall that the Fourier transform of f is:

$$\widehat{f}(\omega) = \langle f, e_\omega \rangle = \int_{\mathbb{R}} f(t)e^{-i\omega t}\, dt\,, \quad e_\omega(t) = e^{i\omega t}$$

It is not a perfect analogue for the time frequency transform Φ since $e_\omega \notin \mathbf{L}^2(\mathbb{R})$, but both transforms analyze f by testing the signal against a family of waveforms. Let us now explore time-frequency transforms in more detail.

Recall the definitions of the time mean u, frequency mean ξ, time variance σ_t^2, and frequency variance σ_ω^2 of a function $f \in \mathbf{L}^2(\mathbb{R})$, first defined when we studied the uncertainty principle in Section 2.4. Apply them to the dictionary $\mathcal{D}$ for each time frequency atom ϕ_γ, and denote the corresponding quantities by

$$u_\gamma, \ \omega_\gamma, \ \sigma_t(\gamma), \ \sigma_\omega(\gamma)$$

The waveform ϕ_γ is essentially supported in time on an interval of length $\sigma_t(\gamma)$, centered at u_γ, while its Fourier transform $\widehat{\phi}_\gamma$ is essentially supported in frequency on an interval

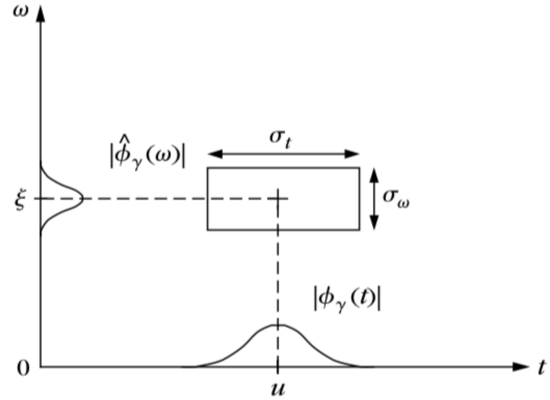

Figure 7: Heisenberg box representing the essential time frequency support of ϕ_γ

of length $\sigma_\omega(\gamma)$, centered at ξ_γ. Thus the joint time frequency support of ϕ_γ in the time frequency plane (t, ω) is given by a Heisenberg box centered at (u_γ, ξ_γ) having time width $\sigma_t(\gamma)$ and frequency width $\sigma_\omega(\gamma)$; see Figure 7.

The Parseval formula (Theorem 2.12) proves that:

$$\Phi f(\gamma) = \int_{\mathbb{R}} f(t)\phi_\gamma^*(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)\widehat{\phi}_\gamma^*(\omega)\,d\omega$$

Thus we see that $\Phi f(\gamma)$ only depends upon the values of $f(t)$ and $\widehat{f}(\omega)$ in the Heisenberg box of ϕ_γ. In particular, $\Phi f(\gamma)$ only measures the frequencies of f in a neighborhood of ξ_γ, and it only measures these frequencies in a neighborhood of the time u_γ. Because of the uncertainty principle (Theorem 2.18), we know that

$$\sigma_t(\gamma)\sigma_\omega(\gamma) \geq \frac{1}{2}$$

Thus it is impossible to measure precisely the frequency response $\widehat{f}(\omega_0)$ at the time t_0. The best we can do is measure the time frequency response of f in a Heisenberg box of area 1/2. Theorem 2.18 proves that the time frequency atoms that achieve this optimal time frequency localization are given by Gabor functions; we will come back to this point shortly when we introduce the windowed Fourier transform.

For pattern recognition and machine learning tasks, it often important to construct time frequency representations that behave well with respect to translations of the signal $f(t)$ (and in 2D, rotations as well). Define $f_u(t) = f(t-u)$ as the translation of f by u, and notice that:

$$\Phi f_u(\gamma) = \int_{\mathbb{R}} f(t-u)\phi_\gamma^*(t)\,dt = \int_{\mathbb{R}} f(t)\phi_\gamma^*(t+u)\,dt = \langle f, \phi_{-u,\gamma}\rangle,$$

where $\phi_{u,\gamma}(t) = \phi_\gamma(t-u)$. This motivates the construction of translation invariant dictionaries. A translation invariant dictionary is obtained by starting with a family of generators

$\{\phi_\gamma\}_{\gamma\in\Gamma}$, and augmenting this family with all translates of each time frequency atom ϕ_γ:

$$\mathcal{D} = \{\phi_{u,\gamma}\}_{u\in\mathbb{R},\gamma\in\Gamma}$$

Set:

$$\overline{\phi}_\gamma(t) = \phi_\gamma^*(-t)$$

The resulting time frequency transform with a translation invariant dictionary is given by:

$$\Phi f(u,\gamma) = \langle f, \phi_{u,\gamma}\rangle = \int_{\mathbb{R}} f(t)\phi_\gamma^*(t-u)\,dt = f * \overline{\phi}_\gamma(u)$$

It thus corresponds to a filtering of f by the time-frequency waveforms $\{\overline{\phi}_\gamma\}_{\gamma\in\Gamma}$.

Exercise 24. Read Section 4.1 of *A Wavelet Tour of Signal Processing*.

4.2 Windowed Fourier Transform

Section 4.2 of A Wavelet Tour of Signal Processing [1].

The Fourier transform $\widehat{f}(\omega)$ tells us every frequency in the signal $f(t)$, but it does not tell us *when* such frequencies are present. For example, in music we hear the time variation of the sound frequencies. Similarly, images with vastly different patterns in them may correspond to different frequencies, localized not over time but space; see the picture of the castle in Figure 8 for an example.

A natural way to account for these localized structures is to localize the Fourier transform with a window function. Let g be a real symmetric window $g(t) = g(-t)$, which has support localized around $t = 0$ (e.g., a Gaussian $g(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-t^2/2\sigma^2}$). Translations of this window by $u \in \mathbb{R}$, and modulations of this window by the frequency $\xi \in \mathbb{R}$, yield a *Gabor type dictionary*:

$$\mathcal{D} = \{g_{u,\xi}\}_{u,\xi\in\mathbb{R}}, \quad g_{u,\xi}(t) = g(t-u)e^{i\xi t}$$

The window is normalized so that $\|g\|_2 = 1$, which implies that $\|g_{u,\xi}\|_2 = 1$ for all $(u,\xi) \in \mathbb{R}^2$. The resulting *windowed Fourier transform* (also known as the short time Fourier transform, or Gabor transform) is:

$$Sf(u,\xi) = \langle f, g_{u,\xi}\rangle = \int_{\mathbb{R}} f(t)g(t-u)e^{-i\xi t}\,dt$$

Notice that $Sf(u,\xi)$ computes a localized version of the Fourier transform of $f(t)$, in which the Fourier integral is localized around u by the window $g(t-u)$.

The energy density of the windowed Fourier transform is the *spectrogram*:

$$P_S f(u,\xi) = |Sf(u,\xi)|^2 = \left|\int_{\mathbb{R}} f(t)g(t-u)e^{-i\xi t}\,dt\right|^2$$

Figure 8: Picture of a castle, taken from Wikipedia. Different regions of the picture have different patterns, such as the sky, the trees, and the castle itself. These patterns have different frequency responses, which are spatially localized.

The spectrogram removes the phase of $Sf(u,\xi)$ and measures the energy of f in a time frequency neighborhood of (u,ξ) specified by the Heisenberg box of $g_{u,\xi}$. The size of these Heisenberg boxes is in fact independent of (u,ξ), as we now show.

First note that since $g(t)$ is even, $g_{u,\xi}$ is centered at u. The variance around u is:

$$\sigma_t^2 = \int_{\mathbb{R}} (t-u)^2 |g_{u,\xi}(t)|^2 \, dt = \int_{\mathbb{R}} t^2 |g(t)|^2 \, dt$$

The Fourier transform $\widehat{g}$ of g is real and symmetric because g is real and symmetric. We also compute the Fourier transform of $g_{u,\xi}$ as (set $e_\xi(t) = e^{i\xi t}$):

$$\begin{aligned} \widehat{g}_{u,\xi}(\omega) &= \widehat{g_u \cdot e_\xi}(\omega) \\ &= (2\pi)^{-1} \widehat{g}_u * \widehat{e}_\xi(\omega) \\ &= (2\pi)^{-1} (e_{-u} \cdot \widehat{g}) * 2\pi \delta_\xi(\omega) \\ &= e^{-iu(\omega-\xi)} \widehat{g}(\omega - \xi) \end{aligned}$$

It follows that $\widehat{g}_{u,\xi}$ is centered at ξ, and

$$\sigma_\omega^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (\omega-\xi)^2 |\widehat{g}_{u,\xi}(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{\mathbb{R}} \omega^2 |\widehat{g}(\omega)|^2 \, d\omega$$

These calculations show that the Heisenberg boxes of $g_{u,\xi}$ centered at (u,ξ) with an area $\sigma_t \sigma_\omega$ that is independent of the location (u,ξ). Thus the windowed Fourier transform has

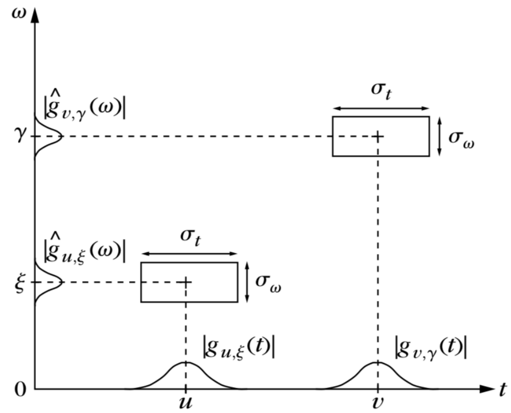

Figure 9: Heisenberg boxes of the windowed Fourier time frequency atoms.

the same resolution across the time frequency plane; see Figure 9. This is one of its defining properties; other time-frequency transforms that we will encounter (e.g., wavelets), will utilize Heisenberg boxes of different dimensions depending on their location in the time-frequency plane.

Exercise 25. Read Section 4.2 of *A Wavelet Tour of Signal Processing*, up to but not including Section 4.2.1.

4.2.1 Parseval for Windowed Fourier

The approach in this section and the next section follows the treatment in [5, Chapter 3]. For a more in depth treatment of the windowed Fourier transform and time frequency analysis, [5] is an excellent resource.

Recall that for a window $g \in \mathbf{L}^2(\mathbb{R})$ the windowed Fourier transform of $f \in \mathbf{L}^2(\mathbb{R})$ is defined as:

$$S_g f(u, \xi) = \int_{\mathbb{R}} f(t) g(t-u) e^{-i\xi t}\, dt$$

Here we write $S_g f$ rather than Sf to emphasize the dependence upon the window choice g. Up till now we have been a little sloppy in that, while we know the windowed Fourier transform is well defined pointwise, we do not know if this transform maps f into some nice functional class. To that end, the next theorem is an analogue of Parseval's formula (Theorem 2.12) for the windowed Fourier transform and for windows g in a subclass of $\mathbf{L}^2(\mathbb{R})$. It shows that $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$. Like the original Parseval formula it is also extremely useful.

Theorem 4.1. *Let $f, h \in \mathbf{L}^2(\mathbb{R})$ and let g be a real symmetric function with $g \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ and $\|g\|_2 = 1$. Then:*

$$\langle f, h \rangle = \frac{1}{2\pi} \langle S_g f, S_g h \rangle_{\mathbf{L}^2(\mathbb{R}^2)}$$

Proof. We will need another fundamental result from real analysis, which is Young's inequality. Suppose that $f_1 \in \mathbf{L}^p(\mathbb{R})$, $f_2 \in \mathbf{L}^q(\mathbb{R})$, and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

Then

$$\|f_1 * f_2\|_r \leq \|f_1\|_p \|f_2\|_q \tag{17}$$

Now define f_ξ as $f_\xi(u) = S_g f(u, \xi)$, so that we think of the windowed Fourier transform as a function in u with a parameter ξ. We first show that $f_\xi \in \mathbf{L}^2(\mathbb{R})$ and then compute its Fourier transform. Additionally, set $g_\xi(t) = g(t)e^{i\xi t}$; we can rewrite $f_\xi(u)$ as (using that g is symmetric):

$$\begin{aligned} f_\xi(u) &= \int_{\mathbb{R}} f(t) g(t-u) e^{-i\xi t} \, dt \\ &= e^{-iu\xi} \int_{\mathbb{R}} f(t) g(u-t) e^{i\xi(u-t)} \, dt \\ &= e^{-iu\xi} f * g_\xi(u) \end{aligned}$$

It thus follows, using Young's inequality, that

$$\|f_\xi\|_2 = \|f * g_\xi\|_2 \leq \|g\|_1 \|f\|_2$$

The Fourier transform of f_ξ is computed as:

$$\widehat{f_\xi}(\omega) = \widehat{f}(\omega + \xi) \widehat{g}_\xi(\omega + \xi) = \widehat{f}(\omega + \xi) \widehat{g}(\omega)$$

Let us now compute the inner product between $S_g f$ and $S_g h$. Since $f_\xi, h_\xi \in \mathbf{L}^2(\mathbb{R})$ we can use Parseval's formula and our computation for their Fourier transform to get:

$$\begin{aligned} \frac{1}{2\pi} \langle S_g f, S_g h \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} S_g f(u, \xi) S_g h^*(u, \xi) \, du \, d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_\xi(u) h_\xi^*(u) \, du \right) d\xi \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}(\omega + \xi) \widehat{h}^*(\omega + \xi) |\widehat{g}(\omega)|^2 \, d\omega \, d\xi \end{aligned} \tag{18}$$

We would like to switch the order of integration using Fubini. To do so we need to bound:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi)|\widehat{g}(\omega)|^2 \, d\omega \, d\xi \\
&\qquad = \int_{\mathbb{R}} |\widehat{g}(\omega)|^2 \int_{\mathbb{R}} |\widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi)| \, d\xi \, d\omega \\
&\qquad \leq \int_{\mathbb{R}} |\widehat{g}(\omega)|^2 \left(\int_{\mathbb{R}} |\widehat{f}(\omega+\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\widehat{h}(\omega+\xi)|^2 \, d\xi \right)^{\frac{1}{2}} d\omega \\
&\qquad \leq (2\pi)^2 \|g\|_2^2 \|f\|_2 \|h\|_2 < \infty
\end{aligned}$$

Thus we can apply Fubini and continuing from (18) we have:

$$\begin{aligned}
(18) &= \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{g}(\omega)|^2 \left(\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi) \, d\xi \right) d\omega \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{g}(\omega)|^2 \left(\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi)\widehat{h}^*(\xi) \, d\xi \right) d\omega \\
&= \langle f, h \rangle \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{g}(\omega)|^2 \, d\omega \\
&= \langle f, g \rangle
\end{aligned}$$

□

The windowed Fourier transform can be extended to any real, symmetric window $g \in \mathbf{L}^2(\mathbb{R})$ using a density argument. Using this extension, we can also extend Theorem 4.1 to any real symmetric window $g \in \mathbf{L}^2(\mathbb{R})$.

Corollary 4.2. *Let $f, h \in \mathbf{L}^2(\mathbb{R})$ and let g be a real symmetric function with $g \in \mathbf{L}^2(\mathbb{R})$ and $\|g\|_2 = 1$. Then:*

$$\langle f, h \rangle = \frac{1}{2\pi} \langle S_g f, S_g h \rangle_{\mathbf{L}^2(\mathbb{R}^2)}$$

It follows from Theorem 4.1 that $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$ and that it preserves the norm, up to a factor of $\sqrt{2\pi}$. This is the analog of the Plancheral formula; we collect it in the next corollary.

Corollary 4.3. *Let $g \in \mathbf{L}^2(\mathbb{R})$. The windowed Fourier transform is a linear map $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$, and it is also an isometry up to a factor of $\sqrt{2\pi}$:*

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}} \|S_g f\|_{\mathbf{L}^2(\mathbb{R}^2)}$$

Exercise 26. Prove Corollary 4.2.

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.

[3] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.

[4] Yves Meyer. *Wavelets and Operators*, volume 1. Cambridge University Press, 1993.

[5] Karlheinz Gröchenig. *Foundations of Time Frequency Analysis*. Springer Birkhäuser, 2001.