Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 10: The Wavelet Transform

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4.4 Wavelet Transforms

Section 4.3 of A Wavelet Tour of Signal Processing.

The hyperbolic chirp example illustrates that some types of signals require a transform that can vary its scale to account for multiscale characteristics within the signal. Another example is given in Figure 13, in which we have a signal with multiple types of behavior, some over large scales (like the general increasing and decreasing nature of the signal) and others over smaller scales (like the singular points). As we shall see a bit later, a multiscale approach

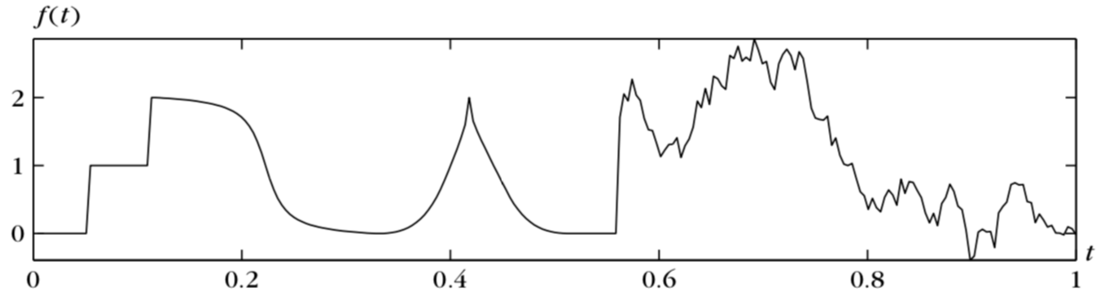

Figure 13: A signal with multiscale structure.

is also necessary for characterizing local singularities in a signal $f(t)$, as it is the ability to "zoom in," which will allow us to achieve such analysis. This is particular important in image processing, since natural images often have many different types of patterns.

A third motivation comes from Remark 3.5. There we defined the operator Δ_j which on the real line would map $f \in \mathbf{L}^2(\mathbb{R})$ to:

$$(\Delta_j f)(t) = \int_{\mathbb{R}} \widehat{f}(\omega) \underbrace{\mathbf{1}_{2^{-j} \leq |\omega| < 2^{-j+1}}(\omega)}_{\widehat{h}_j(\omega)} e^{it\omega} \, d\omega \, .$$

Since the frequency supports of $\widehat{h}_j(\omega)$ decrease as j increases, it must be that the time support of $h_j(t)$ increases as j increases.

All of these examples motivate the introduction of a new multiscale time frequency transform, which will be the wavelet transform. A wavelet $\psi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ is a function with zero average,

$$\int_{\mathbb{R}} \psi(t)\, dt = 0$$

which is well localized in time and frequency. We normalize ψ so that $\|\psi\|_2 = 1$ and such that it is centered at $t = 0$.

A dictionary of time frequency atoms is obtained by dilating ψ by s and translating it by u:

$$\mathcal{D} = \{\psi_{u,s}\}_{u\in\mathbb{R},\, s>0}, \quad \psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

Note that $\|\psi_{u,s}\|_2 = 1$ for all (u, s). The wavelet transform of $f \in \mathbf{L}^2(\mathbb{R})$ computes:

$$Wf(u, s) = \langle f, \psi_{u,s}\rangle = \int_{\mathbb{R}} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right) dt$$

The wavelet dictionary is a translation invariant dictionary, and hence can be written as a family of convolutions. Set

$$\overline{\psi}_s(t) = \frac{1}{\sqrt{s}}\psi^*\left(\frac{-t}{s}\right)$$

and observe that

$$Wf(u, s) = f * \overline{\psi}_s(u)$$

If we set $f_s(u) = Wf(u, s)$, then:

$$\widehat{f_s}(\omega) = \widehat{f}(\omega)\widehat{\overline{\psi}}_s(\omega), \quad \widehat{\overline{\psi}}_s(\omega) = \sqrt{s}\widehat{\psi}^*(s\omega)$$

Since ψ has zero average, the support $\widehat{\psi}(\omega)$ must be away from $\omega = 0$. It follows that the wavelet transform computes a *bandpass* filtering of f with a family dilated bandpass filters $\overline{\psi}_s$.

Wavelets can be either real valued or complex valued. Often in the latter case they are taken to be complex analytic or nearly complex analytic. Such wavelet transforms are good for analyzing instantaneous frequencies, which we studied previously with the windowed Fourier transform. Real valued wavelets, on the other hand, are good for detecting sharp transitions in a signal, such as singular points. In this case the wavelets are defined as multiscale derivative operators. We will start with complex analytic wavelets since they will parallel the windowed Fourier transform to a certain degree, and then study real valued wavelets.

Exercise 37. Read the first part of Section 4.3 of *A Wavelet Tour of Signal Processing*, up to (but not including) Section 4.3.1.

4.4.1 Analytic Wavelet Transform

Section 4.3.2 of A Wavelet Tour of Signal Processing

An analytic wavelet is a complex valued wavelet $\psi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ such that

$$\widehat{\psi}(\omega) = 0, \quad \omega \leq 0$$

We can measure the Heisenberg boxes of the wavelet time frequency atom $\psi_{u,s}$. Suppose that ψ is centered at $t = 0$ so that its time variance is:

$$\sigma_t^2 = \int_{\mathbb{R}} t^2 |\psi(t)|^2 \, dt$$

Since $\psi(t)$ is centered at zero, $\psi_{u,s}(t)$ is centered at $t = u$ and its time variance is computed as:

$$\begin{aligned} \sigma_t^2(u,s) &= \int_{\mathbb{R}} (t-u)^2 |\psi_{u,s}(t)|^2 \, dt \\ &= \int_{\mathbb{R}} (t-u)^2 \frac{1}{s} \left| \psi\left(\frac{t-u}{s}\right) \right|^2 \, dt, \quad v = \frac{t-u}{s} \\ &= s^2 \int_{\mathbb{R}} v^2 |\psi(v)|^2 \, dv \\ &= s^2 \sigma_t^2 \end{aligned}$$

Since ψ is analytic, the center frequency η of ψ is

$$\eta = \frac{1}{2\pi} \int_0^{+\infty} \omega |\widehat{\psi}(\omega)|^2 \, d\omega$$

The Fourier transform of $\psi_{u,s}$ is

$$\sqrt{s}\widehat{\psi}(s\omega) e^{-iu\omega}$$

and thus its center frequency is

$$\xi_{u,s} = \eta / s$$

The frequency variance of ψ is

$$\sigma_\omega^2 = \frac{1}{2\pi} \int_0^{+\infty} (\omega - \eta)^2 |\widehat{\psi}(\omega)|^2 \, d\omega$$

and using a similar change of variables as before, we obtain:

$$\sigma_\omega^2(u,s) = \frac{\sigma_\omega^2}{s^2}$$

Thus the time frequency Heisenberg box of $\psi_{u,s}$ is centered at $(u, \eta/s)$ and has length $s\sigma_t$ along the time axis, and length σ_ω / s along the frequency axis. Figure 14 illustrates the

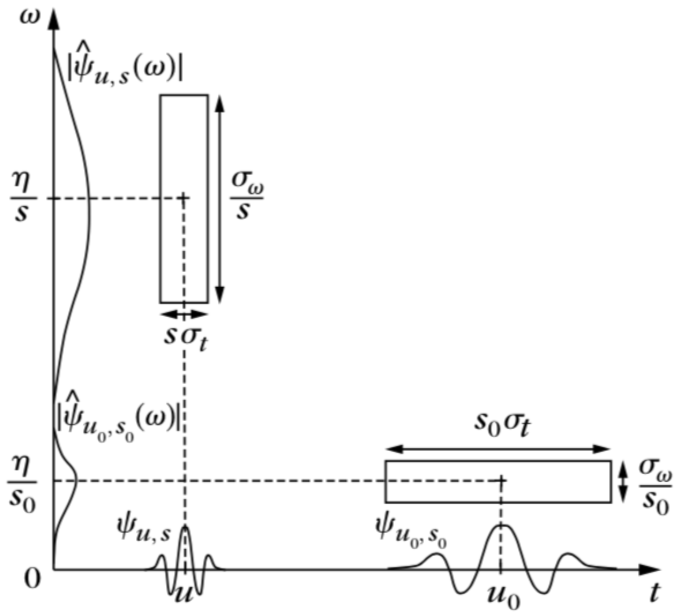

Figure 14: Heisenberg boxes of the analytic wavelet time frequency atoms.

idea. A wavelet transform is thus multiscale in time, as it tests the signal f against localized oscillating waveforms at different scales s. It is additionally, though, multiresolution in frequency, with better frequency localization in the low frequencies, and worse frequency resolution in the high frequencies.

Analogous to the spectrogram for the windowed Fourier transform, an analytic wavelet transform defines a local time frequency energy density $P_W f$, which measures the energy of f in the Heisenberg box $\psi_{u,s}$. This density is called the *scalogram*, and it is defined as:

$$P_W f(u,s) = |Wf(u,s)|^2$$

An analytic wavelet can be constructed in a similar fashion as the windowed Fourier transform. Let g once gain be a real symmetric window, and set

$$\psi(t) = g(t)e^{i\eta t}$$

Recall that Fourier transform of ψ is:

$$\widehat{\psi}(\omega) = \widehat{g}(\omega - \eta)$$

Thus if

$$\widehat{g}(\omega) = 0, \quad |\omega| > \eta$$

then ψ is analytic; see Figure 15. Recall as well that since g is real and symmetric, $\widehat{g}(\omega)$ is real and symmetric as well and thus the center of frequency of ψ is η.

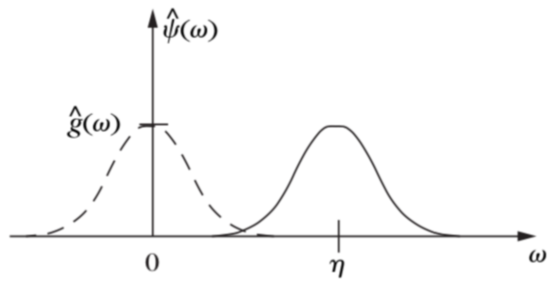

Figure 15: Fourier transform $\widehat{\psi}(\omega)$ of a wavelet $\psi(t) = g(t)e^{i\eta t}$.

A *Gabor wavelet* $\psi(t) = g_\sigma(t)e^{i\eta t}$ is obtained with a Gaussian window

$$g_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$$

where here g_σ is normalized so that $\|\psi\|_1 = 1$. The Fourier transform of the Gaussian window is

$$\widehat{g}_\sigma(\omega) = e^{-\sigma^2\omega^2/2}$$

Thus if $\sigma^2\eta^2 \gg 1$, then $\widehat{g}_\sigma(\omega) \approx 0$ for $|\omega| > \eta$ and the Gabor wavelet has nearly zero average and is nearly analytic. It is thus not a wavelet in the strict sense of the term.

A *Morlet wavelet* modifies the Gabor wavelet to have precisely zero average; it is defined as:

$$\psi(t) = g(t)\left(e^{i\eta t} - C_{\sigma,\eta}\right), \quad C_{\sigma,\eta} = e^{-\sigma^2\eta^2/2}$$

A Morlet wavelet is thus a wavelet since $\int \psi = 0$. It is nearly analytic, but has a small negative response in the negative frequencies. Both 2D Gabor and 2D Morlet wavelets are often used in 2D computer vision and image processing tasks.

The next theorem shows that the analytic wavelet transform is invertible so long as the wavelet ψ satisfies a weak admissibility condition given by (25). Recall that $f_a(t)$ is the analytic part of a real valued signal $f(t)$.

Theorem 4.6. *Let $f \in \mathbf{L}^2(\mathbb{R})$ be real valued and let $\psi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ be an analytic wavelet. Then*

$$Wf(u,s) = \frac{1}{2} Wf_a(u,s) \tag{24}$$

Furthermore, if

$$C_\psi = \int_0^{+\infty} \frac{|\widehat{\psi}(\omega)|^2}{\omega}\, d\omega < \infty \tag{25}$$

then

$$f(t) = \frac{2}{C_\psi} \Re\left[\int_0^{+\infty} \int_{\mathbb{R}} Wf(u,s)\psi_s(t-u)\, du\, \frac{ds}{s^2}\right] \tag{26}$$

and

$$\|f\|_2^2 = \frac{2}{C_\psi}\int_0^{+\infty}\int_{\mathbb{R}} |Wf(u,s)|^2\, du\, \frac{ds}{s^2} \tag{27}$$

Proof. First note that the $\mathbf{L}^1(\mathbb{R})$ norm of ψ_s is:

$$\|\psi_s\|_1 = \sqrt{s}\|\psi\|_1$$

which is obtained through a simple change of variables.

We now prove (24) first. Recall that $\overline{\psi}_s(t) = \psi^*(-t)$ and consider

$$f_s(u) = Wf(u,s) = f * \overline{\psi}_s(u)$$

Young's inequality (17) proves that $f_s \in \mathbf{L}^2(\mathbb{R})$:

$$\|f_s\|_2 = \|f * \overline{\psi}_s\|_2 \leq \|f\|_2\|\overline{\psi}_s\|_1 = \sqrt{s}\|\psi\|_1\|f\|_2 < \infty$$

The Fourier transform of f_s is:

$$\widehat{f_s}(\omega) = \widehat{f}(\omega)\widehat{\overline{\psi}}_s(\omega) = \widehat{f}(\omega)\sqrt{s}\widehat{\psi}^*(s\omega)$$

Since $\widehat{\psi}(\omega) = 0$ for $\omega \leq 0$, and $\widehat{f_a}(\omega) = 2\widehat{f}(\omega)$ for $\omega > 0$, we derive that

$$\widehat{f_s}(\omega) = \frac{1}{2}\widehat{f_a}(\omega)\sqrt{s}\widehat{\psi}^*(\omega)$$

which is the Fourier transform of $(1/2)f_a * \overline{\psi}_s(u)$.

Now let us prove (27). We use the Plancheral formula and Tonelli:

$$\begin{aligned}
\frac{2}{C_\psi}\int_0^{+\infty}\int_{\mathbb{R}} |Wf(u,s)|^2\, du\, \frac{ds}{s^2} &= \frac{1}{2C_\psi}\int_0^{+\infty}\int_{\mathbb{R}} |f_a * \overline{\psi}_s(u)|^2\, du\, \frac{ds}{s^2} \\
&= \frac{1}{C_\psi 4\pi}\int_0^{+\infty}\int_{\mathbb{R}} |\widehat{f_a}(\omega)|^2|\widehat{\psi}(s\omega)|^2\, d\omega\, \frac{ds}{s} \\
&= \frac{1}{4\pi}\int_0^{+\infty} |\widehat{f_a}(\omega)|^2 \frac{1}{C_\psi}\int_0^{+\infty} |\widehat{\psi}(s\omega)|^2 \frac{ds}{s}\, d\omega
\end{aligned} \tag{28}$$

Now make the change of variables $\xi = s\omega$, which induces the change of measure $ds = (s/\xi)d\xi$. We have:

$$\begin{aligned}
(28) &= \frac{1}{4\pi}\int_0^{+\infty} |\widehat{f_a}(\omega)|^2 \frac{1}{C_\psi}\int_0^{+\infty} \frac{|\widehat{\psi}(\xi)|^2}{\xi}\, d\xi\, d\omega \\
&= \frac{1}{4\pi}\int_0^{+\infty} |\widehat{f_a}(\omega)|^2\, d\omega \\
&= \frac{1}{2}\|f_a\|_2^2 = \|f\|_2^2
\end{aligned}$$

Finally we prove (26). To do so we prove that

$$f_a(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} Wf_a(u,s)\psi_s(t-u)\,du\,\frac{ds}{s^2} \tag{29}$$

The result will then follow since $Wf_a(u,s) = 2Wf(u,s)$ and $\Re[f_a(t)] = f(t)$.

To prove (29) we write the right hand side of (29) as:

$$h(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{\mathbb{R}} Wf_a(u,s)\psi_s(t-u)\,du\,\frac{ds}{s^2} = \frac{1}{C_\psi} \int_0^{+\infty} f_a * \overline{\psi}_s * \psi_s(t)\,\frac{ds}{s^2}$$

We compute the Fourier transform of $h(t)$, which gives:

$$\begin{aligned}
\widehat{h}(\omega) &= \frac{1}{C_\psi} \int_0^{+\infty} \widehat{f}_a(\omega)\widehat{\overline{\psi}}_s(\omega)\widehat{\psi}_s(\omega)\,\frac{ds}{s^2} \\
&= \frac{\widehat{f}_a(\omega)}{C_\psi} \int_0^{+\infty} \sqrt{s}\widehat{\psi}^*(s\omega)\sqrt{s}\widehat{\psi}(s\omega)\,\frac{ds}{s^2} \\
&= \frac{\widehat{f}_a(\omega)}{C_\psi} \int_0^{+\infty} |\widehat{\psi}(s\omega)|^2 \frac{ds}{s}
\end{aligned}$$

But this is exactly the same integral we encountered in (28), where we proved that it equals C_ψ. Thus we obtain $\widehat{h}(\omega) = \widehat{f}_a(\omega)$. But then we must have $h(t) = f_a(t)$ and the theorem is proved. □

The condition (25) is the *admissibility condition* of the wavelet ψ. Note that to be finite, we must have $\widehat{\psi}(0) = 0$, or equivalently $\int \psi = 0$. This condition is nearly sufficient. If $\widehat{\psi} \in \mathbf{C}^1(\mathbb{R})$ as well, then one can verify that the admissibility condition is satisfied. By Theorem 2.15, if ψ is sufficiently localized in time such that

$$|\psi(t)| \leq \frac{C}{1 + |t|^{n+1+\epsilon}}$$

then we must have $\widehat{\psi} \in \mathbf{C}^1(\mathbb{R})$.

Exercise 38. Read Section 4.3.2 of *A Wavelet Tour of Signal Processing*.

References

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