Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 14 & 15: Wavelet Characterization of Pointwise Regularity

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Lecturer: Matthew Hirn

5.1.3 Regularity Measurements with Wavelets

Section 6.1.3 of A Wavelet Tour of Signal Processing.

We now prove that "zooming in" on the wavelet coefficients of a signal f characterizes the pointwise regularity of f. We utilize a real valued wavelet $\psi \in \mathbf{C}^n(\mathbb{R})$ with n vanishing moments and with derivatives that have fast decay. The latter point means that for any $0 \leq k \leq n$ and $m \in \mathbb{N}$ there exists $C_m \geq 0$ such that

$$|\psi^{(k)}(t)| \leq \frac{C_m}{1 + |t|^m}, \quad \forall\, t \in \mathbb{R}$$

Let $n - 1 < \alpha \leq n$ for some $n \in \mathbb{N}$. Recall that $f(t)$ is Lipschitz $\alpha > 0$ at $v \in \mathbb{R}$ if there exists a degree $n-1$ polynomial $p_v(t)$ and a constant $K_v \geq 0$ such that

$$\forall\, t \in \mathbb{R}, \quad |f(t) - p_v(t)| \leq K_v |t - v|^\alpha$$

Additionally, f is uniformly Lipschitz α on an interval $[a, b]$ if it satisfies the above condition for all $v \in [a, b]$ with a constant K that is independent of v.

Theorem 5.5. *If $f \in \mathbf{L}^2(\mathbb{R})$ is Lipschitz $\alpha \leq n$ at $v \in \mathbb{R}$, then there exits $A > 0$ such that*

$$|Wf(u, s)| \leq A s^{\alpha + 1/2} \left(1 + \left|\frac{u - v}{s}\right|^\alpha\right), \quad \forall\, (u, s) \in \mathbb{R} \times (0, \infty) \tag{38}$$

Conversely if $\alpha < n$ is not an integer and there exist A and $\alpha' < \alpha$ such that

$$\forall\, (u, s) \in \mathbb{R} \times (0, \infty), \quad |Wf(u, s)| \leq A s^{\alpha + 1/2} \left(1 + \left|\frac{u - v}{s}\right|^{\alpha'}\right) \tag{39}$$

then $f(t)$ is Lipschitz α at $t = v$.

Proof. We first prove (38). Since f is Lipschitz α at v, there exists a polynomial $p_v(t)$ of degree $n - 1$ and K such that

$$|f(t) - p_v(t)| \leq K |t - v|^\alpha$$

Recall that since ψ has n vanishing moments, $Wp_v(u,s) = 0$. Therefore:

$$|Wf(u,s)| = \left| \int_{\mathbb{R}} [f(t) - p_v(t)] \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt \right|$$
$$\leq \int_{\mathbb{R}} K|t-v|^\alpha \frac{1}{\sqrt{s}} \left| \psi\left(\frac{t-u}{s}\right) \right| dt$$

Now make the change of variables $x = (t-u)/s$, which induces the change of measure $dt = sdx$,

$$\begin{aligned}
|Wf(u,s)| &\leq K\sqrt{s} \int_{\mathbb{R}} |sx + u - v|^\alpha |\psi(x)| \, dx \\
&\leq K\sqrt{s} \int_{\mathbb{R}} 2^\alpha (|sx|^\alpha + |u-v|^\alpha) |\psi(x)| \, dx \\
&= K2^\alpha s^{\alpha+1/2} \left(\int_{\mathbb{R}} |x|^\alpha |\psi(x)| \, dx + \left| \frac{u-v}{s} \right|^\alpha \int_{\mathbb{R}} |\psi(x)| \, dx \right) \\
&\leq As^{\alpha+1/2} \left(1 + \left| \frac{u-v}{s} \right|^\alpha \right)
\end{aligned}$$

where

$$A = K2^\alpha \max \left(\int_{\mathbb{R}} |x|^\alpha |\psi(x)| \, dx, \|\psi\|_1 \right)$$

and where we used the fact that $|a+b|^\alpha \leq 2^\alpha(|a|^\alpha + |b|^\alpha)$.

Now we prove (39). This is a difficult proof that adapts the Littlewood-Paley approach referenced earlier in the course. Recall the real wavelet inverse formula from Theorem 4.7,

$$f(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_0^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \frac{ds}{s^2} \, du$$

We are going to break up the scale integral into dyadic intervals. Define:

$$\Delta_j(t) = \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{2^j}^{2^{j+1}} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \frac{ds}{s^2} \, du, \quad j \in \mathbb{Z}$$

Note that we have the following Littlewood-Paley type sum:

$$f(t) = \sum_{j \in \mathbb{Z}} \Delta_j(t) \tag{40}$$

Let $\Delta_j^{(k)}(t)$ be the k^{th} order derivative of $\Delta_j(t)$. To prove that f is Lipschitz α at v we need a polynomial $p_v(t)$ of degree $\lfloor \alpha \rfloor$ and a constant K such that $|f(t) - p_v(t)| \leq K|t-v|^\alpha$. We propose

$$p_v(t) = \sum_{k=0}^{\lfloor \alpha \rfloor} \left(\sum_{j \in \mathbb{Z}} \Delta_j^{(k)}(v) \right) \frac{(t-v)^k}{k!} \tag{41}$$

as a candidate. The remainder of the proof is in showing that (41) does the job. Notice that if f is $n-1$ times differentiable at v, then using (40) we have $\sum_j \Delta_j^{(k)}(v) = f^{(k)}(v)$, and we get the jet $J_v f(t) = p_v(t)$. However, we can compute $\Delta_j^{(k)}(t)$ for each $j \in \mathbb{Z}$ even when f is not n times differentiable at v. But then we need to show that $p_v(t)$ is well defined even when f is not n times differentiable at v, and in particular we need to show that $\sum_{j\in\mathbb{Z}} \Delta_j^{(k)}(v)$ is finite. Once we do that, we can think of (41) as a generalization of jets, and in particular the sums $\sum_j \Delta_j^{(k)}(v)$ as generalizations of pointwise derivatives at v.

We first prove that $\sum_j \Delta_j^{(k)}(v)$ is finite by getting appropriate upper bounds on $|\Delta_j^{(k)}(t)|$ for $k \leq \lfloor \alpha \rfloor + 1 \leq n$. To simplify notation, **we let K be a generic constant that may change from line to line, but does not depend on j and t.** Equation (39) plus the fast decay of the wavelet derivatives yield:

$$\begin{aligned}
|\Delta_j^{(k)}(t)| &= \frac{1}{C_\psi}\left|\int_{\mathbb{R}}\int_{2^j}^{2^{j+1}} Wf(u,s)\frac{1}{\sqrt{s}}\frac{d^k}{dt^k}\psi\left(\frac{t-u}{s}\right)\frac{ds}{s^2}\,du\right| \\
&\leq \frac{1}{C_\psi}\int_{\mathbb{R}}\int_{2^j}^{2^{j+1}} |Wf(u,s)|\frac{1}{\sqrt{s}}\frac{1}{s^k}\left|\psi^{(k)}\left(\frac{t-u}{s}\right)\right|\frac{ds}{s^2}\,du \\
&\leq K\int_{\mathbb{R}}\int_{2^j}^{2^{j+1}} s^{\alpha-k}\left(1+\left|\frac{u-v}{s}\right|^{\alpha'}\right)\frac{C_m}{1+|(t-u)/s|^m}\frac{ds}{s^2}\,du \qquad (42)
\end{aligned}$$

Now observe that on the interval $[2^j, 2^{j+1}]$, we have

$$\sup_{s\in[2^j,2^{j+1}]} s^{\alpha-k} = 2^{\alpha-k}2^{j(\alpha-k)} = K2^{j(\alpha-k)}$$

It follows that

$$\begin{aligned}
(42) &\leq K\int_{\mathbb{R}} 2^{j(\alpha-k)}\left(1+\left|\frac{u-v}{2^j}\right|^{\alpha'}\right)\frac{1}{1+|(t-u)/2^j|^m}\left(\int_{2^j}^{2^{j+1}}\frac{ds}{s^2}\right)du \\
&\leq K\int_{\mathbb{R}} 2^{j(\alpha-k)}\left(1+\left|\frac{u-v}{2^j}\right|^{\alpha'}\right)\frac{1}{1+|(t-u)/2^j|^m}\frac{du}{2^j} \qquad (43)
\end{aligned}$$

Now make the change of variables $u' = 2^{-j}(u-t)$ and once again use the inequality $|a+b|^{\alpha'} \leq 2^{\alpha'}(|a|^{\alpha'}+|b|^{\alpha'})$ to arrive at:

$$\begin{aligned}
(43) &= K2^{j(\alpha-k)}\int_{\mathbb{R}}\left(1+\left|u'+\frac{t-v}{2^j}\right|^{\alpha'}\right)\frac{1}{1+|u'|^m}\,du' \\
&\leq K2^{j(\alpha-k)}\int_{\mathbb{R}}\frac{1+2^{\alpha'}|u|^{\alpha'}+2^{\alpha'}|(t-v)/2^j|^{\alpha'}}{1+|u'|^m}\,du' \\
&\leq K2^{j(\alpha-k)}\left[\int_{\mathbb{R}}\frac{1+|u'|^{\alpha'}}{1+|u'|^m}\,du' + \left|\frac{t-v}{2^j}\right|^{\alpha'}\int_{\mathbb{R}}\frac{1}{1+|u'|^m}\,du'\right]
\end{aligned}$$

Choosing $m = \alpha' + 2$ yields:

$$|\Delta_j^{(k)}(t)| \leq K 2^{j(\alpha-k)} \left(1 + \left|\frac{t-v}{2^j}\right|^{\alpha'}\right), \quad \forall\, k \leq \lfloor\alpha\rfloor + 1 \tag{44}$$

At $t = v$, we obtain

$$|\Delta_j^{(k)}(v)| \leq K 2^{j(\alpha-k)}$$

which guarantees a fast decay of $|\Delta_j^{(k)}(v)|$ as $j \to -\infty$ for $k \leq \lfloor\alpha\rfloor$ (i.e., in the small scale regime), because α is not an integer and so $\alpha > \lfloor\alpha\rfloor$.

At large scales, since

$$|Wf(u,s)| = |f * \overline{\psi}_s(u)| \leq \|f\|_2 \|\psi\|_2$$

with the change variables $u' = (t-u)/s$ we have

$$\begin{aligned}
|\Delta_j^{(k)}(t)| &\leq \frac{1}{C_\psi} \int_{\mathbb{R}} \int_{2^j}^{2^{j+1}} |Wf(u,s)| \frac{1}{\sqrt{s}} \left| \frac{d^k}{dt^k} \psi\left(\frac{t-u}{s}\right) \right| \frac{ds}{s^2}\, du \\
&\leq \frac{\|f\|_2 \|\psi\|_2}{C_\psi} \int_{\mathbb{R}} \int_{2^j}^{2^{j+1}} |\psi^{(k)}(u')| \frac{ds}{s^{3/2+k}}\, du' \\
&\leq \frac{K \|f\|_2 \|\psi\|_2 \|\psi^{(k)}\|_1}{C_\psi} 2^{-j(k+1/2)}
\end{aligned}$$

and therefore

$$|\Delta_j^{(k)}(v)| \leq K 2^{-j(k+1/2)}$$

Thus we can bound $\sum_j \Delta_j^{(k)}(v)$ since

$$\begin{aligned}
\forall\, k \leq \lfloor\alpha\rfloor, \quad \left|\sum_{j\in\mathbb{Z}} \Delta_j^{(k)}(v)\right| &\leq \sum_{j\in\mathbb{Z}} |\Delta_j^{(k)}(v)| \\
&= \sum_{j=-\infty}^{0} |\Delta_j^{(k)}(v)| + \sum_{j=1}^{+\infty} |\Delta_j^{(k)}(v)| \\
&\leq K \sum_{j=-\infty}^{0} 2^{j(\alpha-k)} + K \sum_{j=1}^{+\infty} 2^{-j(k+1/2)} \\
&< +\infty
\end{aligned}$$

With the Littlewood-Paley sum (40) we compute:

$$\begin{aligned}
|f(t) - p_v(t)| &= \left| \sum_{j\in\mathbb{Z}} \left(\Delta_j(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right) \right| \\
&\leq \sum_{j\in\mathbb{Z}} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \qquad (45)
\end{aligned}$$

The sum over the scales $j \in \mathbb{Z}$ is divided in two at J such that

$$2^{J-1} \leq |t - v| \leq 2^J$$

We define:

$$I_J = \sum_{j \geq J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right|$$
$$II_J = \sum_{j < J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right|$$

Note this means our constant K cannot depend on J, since J depends on t. The summands of (45) are $\lfloor \alpha \rfloor$ Taylor approximations of $\Delta_j(t)$ around v. For the large scales corresponding to $j \geq J$, we can use the classical Taylor's theorem to get a bound:

$$\begin{aligned} I_J &= \sum_{j \geq J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \\ &\leq \sum_{j \geq J} \frac{|t-v|^{\lfloor \alpha \rfloor + 1}}{(\lfloor \alpha \rfloor + 1)!} \sup_{h \in [t,v]} |\Delta_j^{\lfloor \alpha \rfloor + 1}(h)| \\ &\leq K |t-v|^{\lfloor \alpha \rfloor + 1} \sum_{j \geq J} \sup_{h \in [t,v]} |\Delta_j^{\lfloor \alpha \rfloor + 1}(h)| \end{aligned}$$

Inserting the bound (44) yields:

$$I_J \leq K|t-v|^{\lfloor \alpha \rfloor + 1} \sum_{j \geq J} 2^{-j(\lfloor \alpha \rfloor + 1 - \alpha)} \left(1 + \left| \frac{t-v}{2^j} \right|^{\alpha'} \right)$$

Since $|t - v| \leq 2^J$ we have $|(t-v)/2^j| \leq 1$ for $j \geq J$. Therefore we have arrived at:

$$I_J \leq K|t-v|^{\lfloor \alpha \rfloor + 1} \sum_{j \geq J} 2^{-j(\lfloor \alpha \rfloor + 1 - \alpha)}$$

We need to bound the series. We will do so with the following proposition.

Proposition 5.6. *For any $0 \leq r < 1$ and for any $J \in \mathbb{Z}$,*

$$\sum_{j \geq J} r^j = \frac{r^J}{1-r}$$

Proof of the Proposition 5.6. The proof will rely on the well known fact:

$$\forall\, R \neq 1\,, \quad \sum_{j=0}^{K-1} R^j = \frac{1 - R^K}{1 - R}$$

which gives the result for $J = 0$ by taking $R = r$ and $K = +\infty$. For $J > 0$ we have

$$\sum_{j \geq J} r^j = \sum_{j \geq 0} r^j - \sum_{j=0}^{J-1} r^j = \frac{1}{1-r} - \frac{1-r^J}{1-r} = \frac{r^J}{1-r}$$

For $J < 0$ we have:

$$\begin{aligned}
\sum_{j \geq J} r^j &= \sum_{j \geq 0} r^j + \sum_{j=J}^{-1} r^j = \frac{1}{1-r} + \sum_{j=1}^{-J} r^{-j} = \frac{1}{1-r} + \sum_{j=1}^{-J} (r^{-1})^j \\
&= \frac{1}{1-r} - 1 + \sum_{j=0}^{-J} (r^{-1})^j = \frac{r}{1-r} + \frac{1-(r^{-1})^{-J+1}}{1-r^{-1}} \\
&= \frac{r}{1-r} + \frac{1-r^{J-1}}{1-r^{-1}} = \frac{r}{1-r} + \frac{r-r^J}{r-1} = \frac{r^J}{1-r}
\end{aligned}$$

□

Now apply Proposition 5.6 with $r_\alpha = 2^{-(\lfloor \alpha \rfloor + 1 - \alpha)}$, which is less than one since $\lfloor \alpha \rfloor + 1 > \alpha$. We obtain:

$$I_J \leq K|t-v|^{\lfloor \alpha \rfloor + 1} \sum_{j \geq J} 2^{-j(\lfloor \alpha \rfloor + 1 - \alpha)} = K|t-v|^{\lfloor \alpha \rfloor + 1} \sum_{j \geq J} r_\alpha^j = K|t-v|^{\lfloor \alpha \rfloor + 1} \frac{r_\alpha^J}{1-r_\alpha}$$

Now $1/(1-r_\alpha)$ only depends on α and can be absorbed into K. Furthermore, since $2^{-J} \leq |t-v|^{-1}$ we also have:

$$r_\alpha^J = 2^{-J(\lfloor \alpha \rfloor + 1 - \alpha)} \leq |t-v|^{-(\lfloor \alpha \rfloor + 1 - \alpha)}$$

Thus we obtain:

$$I_J \leq K|t-v|^{\lfloor \alpha \rfloor + 1} r_\alpha^J \leq K|t-v|^{\lfloor \alpha \rfloor + 1} |t-v|^{-(\lfloor \alpha \rfloor + 1 - \alpha)} = K|t-v|^\alpha$$

Now consider the sum over $j < J$ and use (44),

$$
\begin{aligned}
II_J &= \sum_{j<J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \\
&\leq K \sum_{j<J} \left[|\Delta_j(t)| + \sum_{k=0}^{\lfloor \alpha \rfloor} |\Delta_j(v)| \frac{|t-v|^k}{k!} \right] \\
&\leq K \sum_{j<J} \left[2^{j\alpha} \left(1 + \left| \frac{t-v}{2^j} \right|^{\alpha'} \right) + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|t-v|^k}{k!} 2^{j(\alpha-k)} \right] \\
&= K \sum_{j<J} \left[2^{j\alpha} + 2^{j(\alpha-\alpha')} |t-v|^{\alpha'} + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|t-v|^k}{k!} 2^{j(\alpha-k)} \right] \\
&= K \left[\sum_{j<J} 2^{j\alpha} + |t-v|^{\alpha'} \sum_{j<J} 2^{j(\alpha-\alpha')} + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|t-v|^k}{k!} \sum_{j<J} 2^{j(\alpha-k)} \right] \\
&\leq K \left[\sum_{j \geq -J} (2^{-\alpha})^j + |t-v|^{\alpha'} \sum_{j \geq -J} (2^{\alpha'-\alpha})^j + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|t-v|^k}{k!} \sum_{j \geq -J} (2^{k-\alpha})^j \right] \qquad (46)
\end{aligned}
$$

Apply Proposition 5.6 to each of the series in j with $r_\alpha = 2^{-\alpha} < 1$, $r_\alpha = 2^{\alpha'-\alpha} < 1$, and $r_\alpha = 2^{k-\alpha} < 1$, respectively, to obtain:

$$
(46) \leq K \left[2^{J\alpha} + 2^{J(\alpha-\alpha')} |t-v|^{\alpha'} + \sum_{k=0}^{\lfloor \alpha \rfloor} 2^{J(\alpha-k)} \frac{|t-v|^k}{k!} \right] \qquad (47)
$$

Now, since $2^{J-1} \leq |t-v| \leq 2^J$ we have

$$
2^J = 2 \cdot 2^{J-1} \leq 2|t-v|
$$

Plugging this inequality into (47), we have:

$$
\begin{aligned}
(47) &\leq K \left[2^{\alpha}|t-v|^{\alpha} + 2^{(\alpha-\alpha')}|t-v|^{\alpha-\alpha'}|t-v|^{\alpha'} + \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{|t-v|^k}{k!} 2^{\alpha-k}|t-v|^{\alpha-k} \right] \\
&\leq K|t-v|^{\alpha} + |t-v|^{\alpha} \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{2^{\alpha-k}}{k!} \\
&\leq K|t-v|^{\alpha}
\end{aligned}
$$

Therefore we have $II_J \leq K|t-v|^{\alpha}$ for any J. As a result,

$$
\forall\, t \in \mathbb{R}, \quad |f(t) - p_v(t)| \leq I_J + II_J \leq K|t-v|^{\alpha}
$$

which proves that f is Lipschitz α at v. □

Recall that we are utilizing a real valued wavelet $\psi \in \mathbf{C}^n(\mathbb{R})$ with n vanishing moments and with derivatives that have fast decay. We first remark that the proof of Theorem 5.5 can be adapted to give the following theorem, which measures the uniform Lipschitz regularity of f over arbitrary intervals $[a, b]$.

Theorem 5.7. *If $f \in \mathbf{L}^2(\mathbb{R})$ is uniformly Lipschitz $\alpha \leq n$ over $[a, b]$, then there exists $A > 0$ such that*

$$|Wf(u, s)| \leq As^{\alpha+1/2}, \quad \forall\, (u, s) \in [a, b] \times (0, \infty) \tag{48}$$

Conversely, suppose that f is bounded and that $Wf(u, s)$ satisfies (48) for $\alpha < n$ that is not an integer. Then f is uniformly Lipschitz α on $[a + \epsilon, b - \epsilon]$ for any $\epsilon > 0$.

Proof. The proof relies on Theorem 5.5 and modifications of its proof. See pages 211–212 of *A Wavelet Tour of Signal Processing* for the details. □

We now make a few remarks. First, the condition (48) is only meaningful when $s \to 0$, since in general we have

$$|Wf(u, s)| = |\langle f, \psi_{u,s} \rangle| \leq \|f\|_2 \|\psi\|_2$$

which will supersede (48) for large s. Thus the localized regularity of f is measured by zooming in on the points $u \in [a, b]$.

Second, if ψ has exactly n vanishing moments but f is uniformly Lipschitz $\alpha > n$ on $[a, b]$, then $f \in \mathbf{C}^n(a, b)$ and we showed already in (37) that $\lim_{s\to 0} s^{-(n+1/2)} Wf(u, s) = Kf^{(n)}(u)$ with $K \neq 0$. Thus the wavelet coefficients will not decay as $O(s^{\alpha+1/2})$ despite the higher regularity of f.

Finally, for the converse of Theorems 5.5 and 5.7, there is the requirement that $\alpha \notin \mathbb{Z}$. Indeed, the wavelet decay conditions are not sufficient to conclude α-Lipschitz regularity when $\alpha = n \in \mathbb{Z}$. In the case of $[a, b] = \mathbb{R}$ and $\alpha = 1$, the decay (48) is only sufficient to conclude that f is in the Zygmund class, which consists of all bounded, continuous functions for which there exists a constant K such that

$$|f(t + v) + f(t - v) - 2f(t)| \leq K|v|, \quad \forall\, t, v \in \mathbb{R}$$

For more details, see [4, Chapter 6].

Exercise 48. Read Section 6.1.3 of *A Wavelet Tour of Signal Processing.*

Exercise 49. Show that f may be pointwise Lipschitz $\alpha > 1$ at v, while f' is not pointwise Lipschitz $\alpha - 1$ at v. Consider $f(t) = t^2 \cos(1/t)$ at $t = 0$.

Exercise 50. Let $f(t) = |t|^\alpha$. Show that $Wf(u, s) = s^{\alpha+1/2} Wf(u/s, 1)$. Prove that it is not sufficient to measure the decay of $|Wf(u, s)|$ when $s \to 0$ at $u = 0$ in order to compute the Lipschitz regularity of f at $t = 0$.

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