Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 18 & 19: Stochastic Processes

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5.3 Stochastic Processes

References for this section are:

1. *Wavelet Tour of Signal Processing* [1, Section 6.4]: This covers multi-fractals, of which fractional Brownian motion is an example. We will cover fractional Brownian motion, but not general multi-fractals.

2. *Stochastic Calculus for Finance II* [6]: Chapter 1 and 2 are good references for the basics of measure theoretic probability. Chapter 3 is a good reference for constructing the Wiener process and understanding its properties.

3. *Introduction to random fields and scale invariance* [7]: Some additional good information. I am getting the plots from these notes. Even though the focus is on random fields (that is, stochastic processes in which the index variable $t \in \mathbb{R}^d$), there is good info on stochastic processes as well.

The right hand side of the signal in Figure 23 can be modeled as a stochastic process. Many phenomena of interest can be modeled as stochastic processes that are singular almost everywhere, e.g., financial instruments (stocks), heart records, and textures. Knowing the distribution of singularities is important for analyzing the properties of such processes. However, pointwise measurements are not possible because the singularities are not isolated. If the stochastic process is also self-similar, though, wavelet transforms and in particular wavelet zoom through the layers of self-similarity can extract information about the distribution of singularities. We illustrate this concept on fractional Brownian motions, which are statistically singular almost everywhere with the same type of singularity, specified by its Hurst parameter H.

Recall a probability space consists of three things: (i) the set of all outcomes Ω; (ii) the set of all events $\mathcal{F}$, which is a set of sets, and in which each set $A \in \mathcal{F}$ is a subset $A \subseteq \Omega$. We will require that $\mathcal{F}$ be a σ-algebra, meaning that (a) $\emptyset \in \mathcal{F}$; (b) if $A \in \mathcal{F}$, then the complement of A, denoted A^c, is also in $\mathcal{F}$; and (c) if $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i \geq 1} A_i \in \mathcal{F}$. And finally (iii) a probability measure $\mathbb{P} : \mathcal{F} \to [0,1]$ that assigns each $A \in \mathcal{F}$ a probability $\mathbb{P}(A)$. The probability measure must satisfy (a) $\mathbb{P}(\Omega) = 1$; and (b) if $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then

$$\mathbb{P}\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} \mathbb{P}(A_i)$$

The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is the smallest σ-algebra on $\mathbb{R}$ that contains all intervals. Now we can define a random variable.

Definition 5.12. We say X is a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if

$$X : \Omega \to \mathbb{R}$$

and

$$\forall\, B \in \mathcal{B}\,, \quad \{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$$

Definition 5.13. The distribution of a random variable X is the probability measure $\mu_X : \mathcal{B} \to [0,1]$ defined as

$$\mu_X(B) = \mathbb{P}(X \in B)$$

Definition 5.14. A real valued stochastic process $X = (X(t))_{t\in\mathbb{R}}$ is a family of random variables

$$X(t) : \Omega \to \mathbb{R}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We will assume that all of our stochastic processes are real valued.

Definition 5.15. The distribution of a stochastic process X is given by all its finite dimensional distributions, that is, the distribution of all real random vectors

$$(X(t_1), \ldots, X(t_d))\,, \quad \forall\, d \geq 1\,, \;\; \forall\, t_1, t_2, \ldots, t_d$$

Definition 5.16. A stochastic process X is a second order process if $\mathbb{E}[X(t)^2] < +\infty$ for all $t \in \mathbb{R}$. In this case we may define its:

- Mean function

$$m_X(t) = \mathbb{E}[X(t)] = \int_\Omega X(t)(\omega)\, d\mathbb{P}(\omega)$$

  The process is centered if $m_X(t) = 0$ for all $t \in \mathbb{R}$. Also note that, unfortunately, the standard notation in harmonic analysis for the frequency variable is ω, but the standard notation in probability for an outcome is also ω. Hopefully the context will always be clear and things will not be too confusing.

- Covariance function

$$\mathrm{Cov}_X(s,t) = \mathrm{Cov}(X(s), X(t)) = \mathbb{E}[(X(s) - \mathbb{E}[X(s)])(X(t) - \mathbb{E}[X(t)])]$$

  Note that the variance is given by

$$\mathrm{Var}_X(t) = \mathrm{Var}(X(t)) = \mathrm{Cov}(X(t), X(t)) = \mathbb{E}[(X(t) - [X(t)])^2]$$

Definition 5.17. A stochastic process X is Gaussian if for all $t_1, t_2, \ldots, t_d$ the probability distribution of the random vector $(X(t_1), \ldots, X(t_d)) \in \mathbb{R}^d$ is normally distributed.

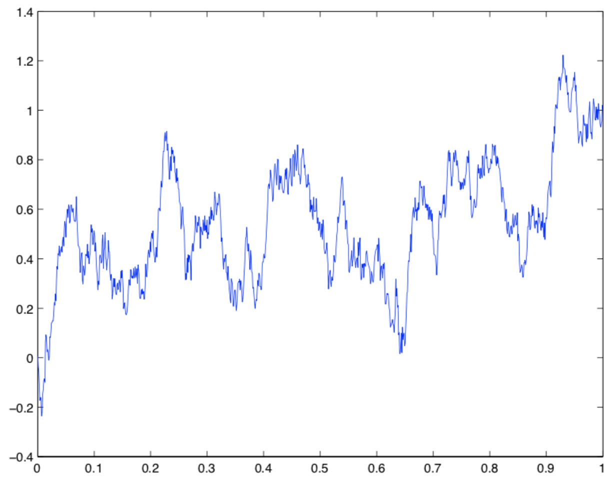

Figure 28: A sample path (realization) of the Wiener process (Brownian motion).

From the properties of the normal distribution it follows that the probability distribution of a Gaussian process is entirely determined by the mean function $m_X(t)$ and the covariance function $\mathrm{Cov}_X(s,t)$. A very useful and famous example of a Gaussian process is the Wiener process (also referred to as Brownian motion), which has wide use in physics and finance and other fields. Let us denote it by $W(t)$. The Wiener process satisfies the following conditions:

- W is a Gaussian process with $W(0) = 0$
- $W(t)$ is continous in t
- $m_W(t) = \mathbb{E}[W(t)] = 0$ for all $t \in \mathbb{R}$
- $\mathrm{Cov}_W(s,t) = \frac{1}{2}(|s| + |t| - |t-s|) = \min(|s|, |t|)$ for all $s, t \in \mathbb{R}$

Figure 28 plots a sample path (that is, a realization) of the Wiener process.

Let us describe now how to construct the Wiener process. It will give us some intuition about stochastic processes in general. The main idea is that we are going to construct a random walk out of an infinite sequence of coin flips. We will then let these coin flips happen with increasing frequency, until in the limit there is no time between the flips. Let us now be more precise.

First consider the experiment of flipping a coin once. There are two possible outcomes, heads or tails, and our probability space is the following:

$$\begin{aligned} \Omega_1 &= \{H, T\} \\ \mathcal{F}_1 &= \{\emptyset, \{H\}, \{T\}, \Omega_2\} \\ \mathbb{P}_1(H) &= p \\ \mathbb{P}_1(T) &= q = 1 - p \end{aligned}$$

Note that $\mathbb{P}_1(\emptyset) = 0$ and $\mathbb{P}_1(\Omega_1) = 1$ by the properties of probability measures. We define a random variable on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ that takes the value $+1$ for the outcome heads, and -1 for the outcome tails:

$$Z(\omega) = \begin{cases} +1 & \omega = H \\ -1 & \omega = T \end{cases} \tag{51}$$

Now let us consider the experiment of flipping a coin infinitely many times, in which all the flips are independent. In this case our set of outcomes is:

$$\Omega_\infty = \text{ all infinite sequences of heads (H) and tails (T)}$$

This outcome space is uncountably infinite, so more care is needed in defining its σ-algebra $\mathcal{F}_\infty$ and its probability measure $\mathbb{P}_\infty$. We will do so by specifying the probability of all events that are based on a finite number of coin tosses. Note that an outcome $\omega \in \Omega_\infty$ can be written as:

$$\omega = \omega_1\omega_2\omega_3\ldots$$

where each $\omega_i \in \{H, T\}$. Now let us build up $\mathcal{F}_\infty$. We know we have to put $\emptyset, \Omega_\infty \in \mathcal{F}_\infty$ with $\mathbb{P}_\infty(\emptyset) = 0$ and $\mathbb{P}_\infty(\Omega_\infty) = 1$. Now let us also add in the two events:

$$\begin{aligned} A_H &= \{\omega \in \Omega_\infty : \omega_1 = H\} = \text{ first coin is a heads} \\ A_T &= \{\omega \in \Omega_\infty : \omega_1 = T\} = \text{ first coin is a tails} \end{aligned}$$

Based on the single coin toss probability space, we set

$$\mathbb{P}_\infty(A_H) = p \quad \text{and} \quad \mathbb{P}_\infty(A_T) = q$$

Remember that if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$, but in this case $A_H^c = A_T$ so we are okay. Also note the union is $A_T \cup A_H = \Omega_\infty$. Now we add in events based on the first two coin tosses, where the definitions of these events should be clear:

$$A_{HH}\,,\, A_{HT}\,,\, A_{TH}\,,\, A_{TT}$$

We set the probabilities accordingly:

$$\begin{aligned} \mathbb{P}_\infty(A_{HH}) &= p^2 \\ \mathbb{P}_\infty(A_{HT}) &= pq \\ \mathbb{P}_\infty(A_{TH}) &= qp \\ \mathbb{P}_\infty(A_{TT}) &= q^2 \end{aligned}$$

Now we have to take these four new events, and also consider their compliments and unions, and also add those events into $\mathcal{F}_\infty$, and specify their probabilities. This can be done. Then we continue by considering events based on the first three coin tosses, then the first four coins tosses, and so on, adding everything into $\mathcal{F}_\infty$ along with unions and compliments, and specifying probabilities. We do this for all events which are based on the first k coin tosses, for all $k \in \mathbb{N}$. Then we complete $\mathcal{F}_\infty$ by adding in the minimal number of all other events required to have a σ-algebra.

Now we define the random walk, which is a discrete stochastic process defined over the probability space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$. We set $p = q = 1/2$, so the coin if fair, which will make the walk unbiased. Let $M = (M(n))_{n\in\mathbb{N}_0}$ be the random walk, where $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$, and where each $M(n)$ is a random variable on $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$. Define

$$M(0)(\omega) = 0\,, \quad \forall\, \omega \in \Omega_\infty$$

That is to say, our random walk will always start at zero. For the remaining steps, recall that $\omega = \omega_1\omega_2\omega_3\ldots$ is an outcome in Ω_∞. Let $Z_i(\omega_i)$ be defined as in (51) for each coin flip ω_i. Define $M(n)$ for every $n \in \mathbb{N}$ as

$$M(n)(\omega) = \sum_{i=1}^{n} Z_i(\omega_i)\,, \quad \omega = \omega_1\omega_2\omega_3\ldots$$

The Wiener process on (Brownian motion) $[0, \infty)$ is obtained by rescaling the random walk M. Define $W^{(m)} = (W^{(m)}(t))_{t\in[0,\infty)}$ as

$$W^{(m)}(t) = \begin{cases} (1/\sqrt{m})M(mt) & mt \in \mathbb{N}_0 \\ (1/\sqrt{m})[(\lceil mt\rceil - mt)M(\lfloor mt\rfloor) + (mt - \lfloor mt\rfloor)M(\lceil mt\rceil)] & mt \notin \mathbb{N}_0 \end{cases}$$

We then obtain $W = (W(t))_{t\in[0,\infty)}$ by taking $m \to \infty$, that is

$$W(t) = \lim_{m\to\infty} W^{(m)}(t)$$

To obtain a Wiener process on $\mathbb{R}$, we take two independent Wiener processes $W_1 = (W_1(t))_{t\in[0,\infty)}$ and $W_2 = (W_2(t))_{t\in[0,\infty)}$ and we create one on $\mathbb{R}$ by setting:

$$W(t) = \begin{cases} W_1(t) & t \geq 0 \\ W_2(-t) & t < 0 \end{cases}$$

The Wiener process inherits the properties of the random walk. In particular, $W(0) = 0$ and for all $t_0 < t_1 < t_2 < \cdots t_k$ the increments

$$W(t_1) - W(t_0)\,, W(t_2) - W(t_1)\,, \ldots, W(t_k) - W(t_{k-1})$$

are independent and each increment is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0$$
$$\mathrm{Var}(W(t_{i+1}) - W(t_i)) = t_{i+1} - t_i$$

The other properties in our original definition also follow from this construction.

Let us now consider another important class of stochastic processes.

Definition 5.18. A stochastic process X is stationary if, for all $u \in \mathbb{R}$, the stochastic process $(X(t+u))_{t\in\mathbb{R}}$ has the same distribution as $X = (X(t))_{t\in\mathbb{R}}$.

The Wiener process is not stationary, but its increments are. We will come back to this point later. For now, we remark that stationary processes are translation invariant since their distribution does not change with a temporal translation by u. Their statistics inherit this invariance, as the following proposition illustrates.

Proposition 5.19. *If a second order stochastic process X is stationary, then*

- *Its mean function is constant, that is $m_X(t) = m_X$ for some constant value m_X. We will sometimes write $m_X = \mathbb{E}[X]$.*
- *Its covariance function only depends on $t - s$, that is*

$$\mathrm{Cov}_X(s,t) = R_X(t-s)$$

  for some even function $R_X : \mathbb{R} \to \mathbb{R}$. The function R_X also satisfies $R_X(0) \geq 0$ and $|R_X(\tau)| \leq R_X(0)$ for all $\tau \in \mathbb{R}$.

Proof. By the stationarity of X we have $X(t) \overset{d}{=} X(0)$ (that is, $X(t)$ and $X(0)$ have the same distribution) and so $m_X(t) = \mathbb{E}[X(t)] = \mathbb{E}[X(0)] = m_X(0)$ for all $t \in \mathbb{R}$. For the covariance set $R_X(\tau) = \mathrm{Cov}_X(0,\tau)$. For any $s \in \mathbb{R}$ we have, again by the stationarity of X, that $(X(s), X(\tau+s)) \overset{d}{=} (X(0), X(\tau))$ and so $\mathrm{Cov}_X(s, \tau+s) = \mathrm{Cov}(0,\tau) = R_X(\tau)$. Hence for $\tau = t - s$ we have $\mathrm{Cov}_X(s,t) = R_X(t-s)$. Since $(X(0), X(\tau)) \overset{d}{=} (X(-\tau), X(0))$ we have

$$R_X(\tau) = \mathrm{Cov}_X(0,\tau) = \mathrm{Cov}_X(-\tau, 0) = \mathrm{Cov}_X(0,-\tau) = R_X(-\tau)$$

and so R_X is even. We also have

$$R_X(0) = \mathrm{Cov}_X(0,0) = \mathrm{Var}(X(0)) \geq 0$$

Finally, using the Cauchy-Schwarz inequality and the stationarity of X:

$$\begin{aligned}
|R_X(\tau)| = |\mathrm{Cov}(X(0), X(\tau))| &\leq \sqrt{\mathrm{Var}(X(0))\mathrm{Var}(X(\tau))} \\
&= \sqrt{\mathrm{Var}(X(0))\mathrm{Var}(X(0))} \\
&= \mathrm{Var}(X(0)) \\
&= R_X(0)
\end{aligned}$$

□

Examples of stationary Gaussian processes are given by Ornstein Uhlenbeck processes, which are defined for any $\theta > 0$ as:

$$X(t) = e^{-\theta t} W(e^{2\theta t})$$

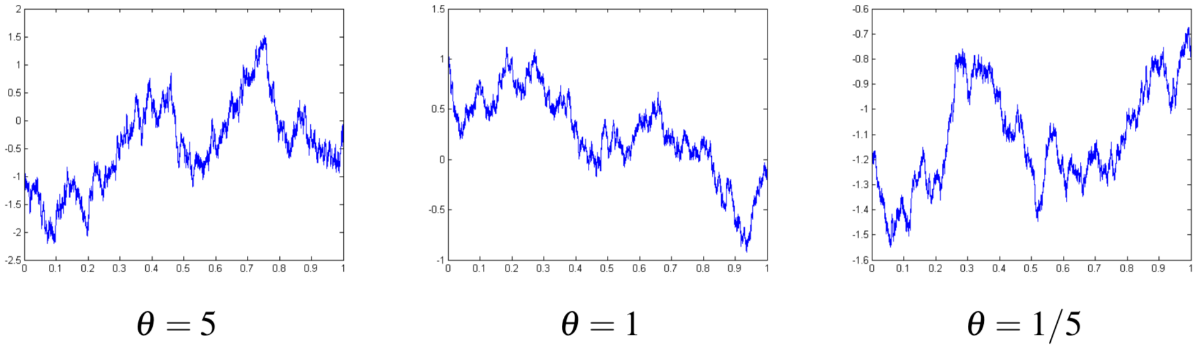

Figure 29: Sample paths of the Ornstein Uhlenbeck process for different values of θ.

where W is the Wiener process. It is clear $\mathbb{E}[X(t)] = 0$ for all $t \in \mathbb{R}$. Also a short calculation shows its covariance is

$$\mathrm{Cov}_X(s,t) = e^{-\theta|t-s|}$$

and thus only depends on $t - s$. Figure 29 plots Ornstein Uhlenbeck processes for different values of θ. Using the previous proposition, we can define the power spectral density (power spectrum) of a stationary process X.

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, 3rd edition, 2008.

[2] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. Princeton Lectures in Analysis. Princeton University Press, 2003.

[3] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.

[4] Yves Meyer. *Wavelets and Operators*, volume 1. Cambridge University Press, 1993.

[5] Karlheinz Gröchenig. *Foundations of Time Frequency Analysis*. Springer Birkhäuser, 2001.

[6] Steven Shreve. *Stochastic Calculus for Finance II*. Springer-Verlag New York, 2004.

[7] Hermine Biermé. Introduction to random fields and scale invariance. hal-01493834, 2018.