Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 20: Time-Frequency Analysis of Stationary Processes

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Definition 5.20. The power spectral density of a second order stationary process X is the Fourier transform of $R_X(\tau)$, that is, $\widehat{R}_X(\omega)$.

For a stationary process X, the function $R_X(\tau) = \mathrm{Cov}_X(0,\tau)$ measures the variability of random fluctuations of X over time. The power spectral density organizes the total variability of X over all times into different frequency components. A time frequency transforms of a stationary process allow us to measure the variability of X within time-frequency Heisenberg boxes. For example, the wavelet coefficients of X define a family of new stochastic processes $X * \psi_s$, indexed by the scale parameter $s > 0$, which are defined as

$$WX(u,s) = X * \psi_s(u) = \int_{\mathbb{R}} X(t)\psi_s(u-t)\,dt$$

We assume $\psi_s(t)$ is continuous, real valued, and compactly supported. Note the integral of a stochastic process with continous sample paths times a continuous deterministic function $f(t)$, over a finite integral, is a random variable defined using the Riemann integral:

$$\int_a^b X(t)f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} X(t_i^n)f(t_i^n)(t_{i+1}^n - t_i^n)$$

where $a = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = b$ for all n, and $\delta_n = \max_{0\leq i\leq n-1} |t_{i+1} - t_i| \to 0$ as $n \to \infty$. The new stochastic process $X * \psi_s = (X * \psi_s(u))_{u\in\mathbb{R}}$ retains only the fluctuations of X at the scale s, for each time u; smaller and larger scale fluctuations are eliminated because the wavelet ψ_s has a frequency support essentially supported in a frequency band determined by the scale s. The next proposition encodes this statement more precisely.

Theorem 5.21. *Let X be a second order stationary process with continuous sample paths and with mean zero, i.e., $\mathbb{E}[X] = 0$, and let ψ be a continuous real valued wavelet with compact support. Then $X * \psi_s$ is a stationary process for each $s > 0$ and:*

$$\widehat{R}_{X*\psi_s}(\omega) = \widehat{R}_X(\omega)|\widehat{\psi}_s(\omega)|^2 = s|\widehat{\psi}(s\omega)|^2\widehat{R}_X(\omega) \tag{52}$$

Proof. The fact that $X*\psi_s$ is stationary is straightforward. We also note that since $\mathbb{E}[X] = 0$, we also have $\mathbb{E}[X * \psi_s] = 0$ for each $s > 0$. Thirdly, if $R_X \in \mathbf{L}^1(\mathbb{R})$ then $R_{X*\psi_s} \in \mathbf{L}^1(\mathbb{R})$;

indeed:

$$
\begin{aligned}
\int_{\mathbb{R}} |R_{X*\psi_s}(\tau)|\, d\tau &= \int_{\mathbb{R}} |\mathbb{E}[X*\psi_s(0)X*\psi_s(\tau)]|\, d\tau \\
&= \int_{\mathbb{R}} \left|\mathbb{E}\left[\int_{\mathbb{R}} X(u)\psi_s(-u)\, du \cdot \int_{\mathbb{R}} X(v)\psi_s(\tau - v)\, dv\right]\right|\, d\tau \\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[X(u)X(v)]\psi_s(-u)\psi_s(\tau - v)\, du\, dv\right|\, d\tau \\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}}\int_{\mathbb{R}} R_X(u-v)\psi_s(-u)\psi_s(\tau - v)\, du\, dv\right|\, d\tau \qquad \text{(CoV: } t = u - v\text{)} \\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}}\int_{\mathbb{R}} R_X(t)\psi_s(-(t+v))\psi_s(\tau - v)\, dt\, dv\right|\, d\tau \\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}} \psi_s(\tau - v)\int_{\mathbb{R}} R_X(t)\psi_s(-v-t)\, dt\, dv\right|\, d\tau \\
&= \int_{\mathbb{R}} \left|\int_{\mathbb{R}} \psi_s(\tau - v) R_X * \psi_s(-v)\, dv\right|\, d\tau \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\psi_s(\tau - v) R_X * \psi_s(-v)|\, dv\, d\tau \\
&= \int_{\mathbb{R}} |R_X * \psi_s(-v)| \int_{\mathbb{R}} |\psi_s(\tau - v)|\, d\tau\, dv \\
&= \|\psi_s\|_1 \|R_X * \psi_s\|_1 \\
&\leq \|R_X\|_1 \|\psi_s\|_1^2
\end{aligned}
$$

Since ψ is continuous and compactly supported, it is in $\mathbf{L}^1(\mathbb{R})$ and so the bound is finite, and $R_{X*\psi_s} \in \mathbf{L}^1(\mathbb{R})$.

Now let us prove (52). Many of the steps are the same as above.

$$
\begin{aligned}
\widehat{R}_{X*\psi_s}(\omega) &= \int_{\mathbb{R}} R_{X*\psi_s}(\tau)e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} \mathbb{E}[X*\psi_s(0)X*\psi_s(\tau)]e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} \mathbb{E}\left[\int_{\mathbb{R}} X(u)\psi_s(-u)\,du \cdot \int_{\mathbb{R}} X(v)\psi_s(\tau-v)\,dv\right] e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[X(u)X(v)]\psi_s(-u)\psi_s(\tau-v)\,du\,dv\right] e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} R_X(u-v)\psi_s(-u)\psi_s(\tau-v)\,du\,dv\right] e^{-i\omega\tau}\,d\tau \qquad (\text{CoV: } t = u-v) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} R_X(t)\psi_s(-(t+v))\psi_s(\tau-v)\,dt\,dv\right] e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \psi_s(\tau-v)\int_{\mathbb{R}} R_X(t)\psi_s(-v-t)\,dt\,dv\right] e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \psi_s(\tau-v)R_X*\psi_s(-v)\,dv\right] e^{-i\omega\tau}\,d\tau \\
&= \int_{\mathbb{R}} R_X*\psi_s(-v)\int_{\mathbb{R}} \psi_s(\tau-v)e^{-i\omega\tau}\,d\tau\,dv \\
&= \widehat{\psi}_s(\omega)\int_{\mathbb{R}} R_X*\psi_s(-v)e^{-i\omega v}\,dv \\
&= \widehat{\psi}_s(\omega)\widehat{R}_x^*(\omega)\widehat{\psi}_s^*(\omega) \\
&= \widehat{R}_X(\omega)|\widehat{\psi}_s(\omega)|^2
\end{aligned}
$$

where the last equality follows from recalling that $R_X(\tau)$ is an even function, and hence its Fourier transform is real valued. □

Stationarity is a pretty strict assumption, and as mentioned, does not include the Wiener process. The notion of a stochastic process with stationary increments relaxes this requirement and includes a much larger number of stochastic processes.

Definition 5.22. A stochastic process X has stationary increments if, for all $u \in \mathbb{R}$, the stochastic process $(X(t+u) - X(u))_{t\in\mathbb{R}}$ has the same distribution as $(X(t) - X(0))_{t\in\mathbb{R}}$.

Stochastic processes with stationary increments include many more processes than just stationary processes, which allows us to model a wider variety of phenomena. Note, in particular, if X has stationary increments then the mean and variance of an increment depends only on the length of the increment, not where it started. That is for any $u \in \mathbb{R}$,

$$
\begin{aligned}
\mathbb{E}[X(t+u) - X(u)] &= \mathbb{E}[X(t) - X(0)] \\
\mathrm{Var}(X(t+u) - X(u)) &= \mathrm{Var}(X(t) - X(0))
\end{aligned}
$$

An example of a stochastic process with stationary increments is the Wiener process.

Theorem 5.23. *The Wiener process, W, has stationary increments.*

Proof. Define the stochastic process $(\widetilde{W}(t))_{t\in\mathbb{R}}$ as

$$\widetilde{W}(t) = W(t+u) - W(u)$$

where $u \in \mathbb{R}$ is fixed but arbitrary. Our goal is to show distribution of $\widetilde{W}$ does not depend on u, which would mean that W has stationary increments. We first note the Wiener process, W, is a Gaussian process, and thus so is $\widetilde{W}$. Therefore, if we can show the mean function and the covariance function of $\widetilde{W}$ do not depend on u then we are finished. For the mean function we have

$$m_{\widetilde{W}}(t) = \mathbb{E}[\widetilde{W}(t)] = \mathbb{E}[W(t+u)] - \mathbb{E}[W(u)] = 0 - 0 = 0$$

which is obviously independent of u. For the covariance function we have:

$$\begin{aligned}
&2\text{Cov}_{\widetilde{W}}(s,t) \\
&\quad = 2\mathbb{E}[\widetilde{W}(s)\widetilde{W}(t)] \\
&\quad = 2\mathbb{E}[(W(s+u) - W(u))(W(t+u) - W(u))] \\
&\quad = 2\mathbb{E}[W(s+u)W(t+u)] + 2\mathbb{E}[W(u)^2] - 2\mathbb{E}[W(u)W(s+u)] - 2\mathbb{E}[W(u)W(t+u)] \\
&\quad = |s+u| + |t+u| - |t-s| + |u| + |u| - |u| - |u+s| + |s| - |u| - |t-u| + |t| \\
&\quad = |t| + |s| - |t-s|
\end{aligned}$$

which is also independent of u. □

Fractional Brownian motion [7, 8] is a generalization of Brownian motion (i.e., the Wiener process). It depends on a parameter H, which is called the Hurst parameter.

Definition 5.24. A stochastic process $B_H = (B_H(t))_{t\in\mathbb{R}}$ is called a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ if it satisfies the following:

- B_H is a Gaussian process with $B_H(0) = 0$
- $B_H(t)$ is continuous in t
- $m_{B_H}(t) = \mathbb{E}[B_H(t)] = 0$ for all $t \in \mathbb{R}$
- $\text{Cov}_{B_H}(s,t) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$ for all $s, t \in \mathbb{R}$.

Notice that when $H = 1/2$ we obtain regular Brownian motion, i.e., the Wiener process. Figure 30 plots three sample paths of fBm for $H = 0.75$, while Figure 31 plots sample paths of fBm for $H = 0.15, 0.55, 0.95$.

First note, that like the Wiener process, fractional Brownian motion has stationary increments for any $H \in (0,1)$. Indeed, the proof is essentially identical.

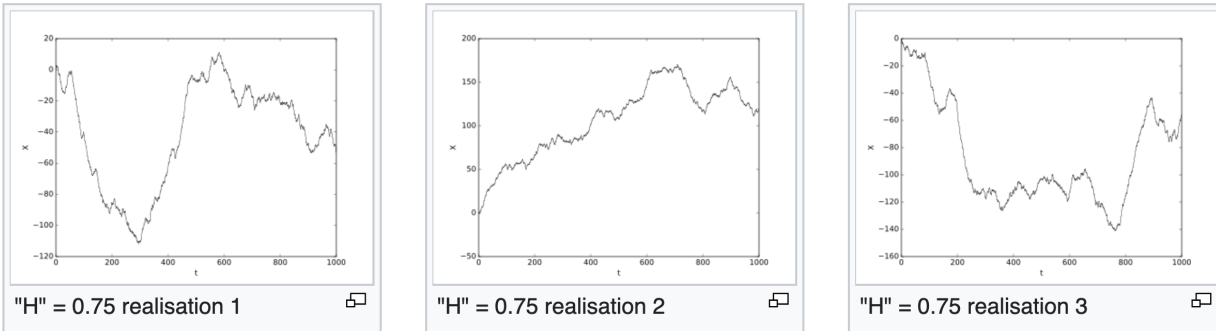

Figure 30: Three sample paths of fractional Brownian motion with Hurst parameter $H = 0.75$. Figure taken from https://en.wikipedia.org/wiki/Fractional_Brownian_motion.

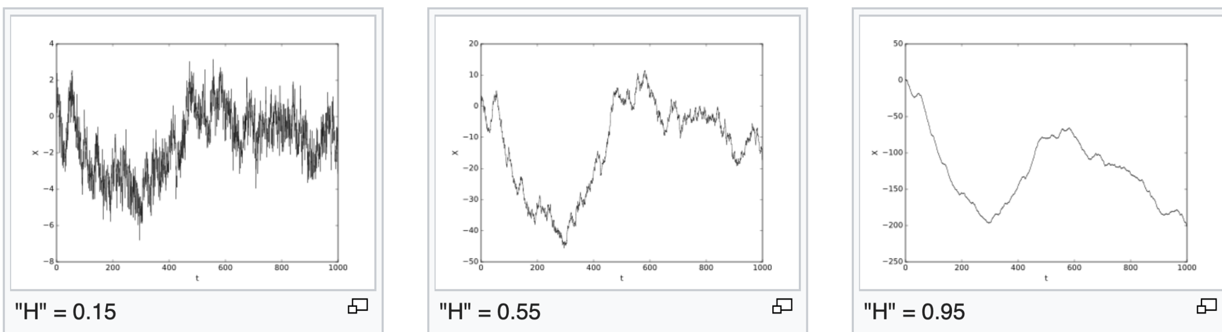

Figure 31: Sample paths of fractional Brownian motion with Hurst parameter $H = 0.15$ (left), $H = 0.55$ (middle), and $H = 0.95$ (right). Figure taken from https://en.wikipedia.org/wiki/Fractional_Brownian_motion.

We also remark that fBm, and hence the Wiener process too, are self similar. A stochastic process $X = (X(t))_{t\in\mathbb{R}}$ is self-similar of order H if

$$\forall\, a > 0\,, \quad (X(at))_{t\in\mathbb{R}} \stackrel{d}{=} a^H (X(t))_{t\in\mathbb{R}}$$

From the definition of fBm we see it is self-similar, as its mean function satisfies

$$\mathbb{E}[B_H(at)] = 0 = \mathbb{E}[a^H B_H(t)]$$

and its covariance function satisfies

$$\begin{aligned}
\mathbb{E}[B_H(as)B_H(at)] &= \frac{1}{2}(|as|^{2H} + |at|^{2H} - |as - at|^{2H}) \\
&= \frac{a^{2H}}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}) \\
&= \mathbb{E}[a^H B_H(s) a^H B_H(t)]
\end{aligned}$$

Furthermore, since it is a Gaussian process, it is completely determined by its mean function and covariance function.

References

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