Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 21 & 22: Time-Frequency Analysis of fBm

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Fractional Brownian motions are interesting because they introduce several new behaviors relative to the Wiener process. For example, recall that the increments of regular Brownian motion, i.e. the Wiener process, are independent. This is not the case for fractional Brownian motion when $H \neq 1/2$. In fact the increments of fBm are negatively correlated for $H \in (0, 1/2)$ and positively correlated for $H \in (1/2, 1)$. To see this let $s_1 < t_1 < s_2 < t_2$ so that the intervals $[s_1, t_1]$ and $[s_2, t_2]$ are non-overlapping, and observe that

$$\begin{aligned}\mathrm{Cov}(B_H(t_1) - B_H(s_1), &B_H(t_2) - B_H(s_2)) \\ &= \mathbb{E}[(B_H(t_1) - B_H(s_1))(B_H(t_2) - B_H(s_2))] \\ &= \frac{1}{2}\left(|t_2 - s_1|^{2H} - |t_2 - t_1|^{2H} - (|s_2 - s_1|^{2H} - |s_2 - t_1|^{2H})\right)\end{aligned}$$

Now note that $t_2 - s_1 - (t_2 - t_1) = t_1 - s_1$ and $s_2 - s_1 - (s_2 - t_1) = t_1 - s_1$ and the function x^{2H} is concave when $H \in (0, 1/2)$ and convex when $H \in (1/2, 1)$. It follows that

$$\mathrm{Cov}(B_H(t_1) - B_H(s_1), B_H(t_2) - B_H(s_2)) \begin{cases} < 0 & H \in (0, 1/2) \\ > 0 & H \in (1/2, 1) \end{cases}$$

Therefore for $H \in (0, 1/2)$ the fBm is counter-persistent. That is, if it was increasing in the past, it is more likely to decrease in the future. On the other hand, for $H \in (1/2, 1)$, fBm is persistent. That is, the past trend is likely to continue in the future. We can this phenomenon in Figure 31, in which the the realization for $H = 0.15$ tends to go up down with a much higher frequency than than the realization for $H = 0.95$.

Visually, it would also appear the realizations become smoother with increasing Hurst parameter H. This is in fact the case, as the modulus of continuity of fractional Brownian motion is [8]:

$$\omega_{B_H}(\delta) = \delta^H |\log \delta|^{1/2}$$

Thus it is nearly H-Hölder, but not quite. Later on we will show the decay of the wavelet coefficients as the scale $s \to 0$ of fractional Brownian motion characterize the Hurst exponent H, and hence the regularity of B_H, even though realizations of B_H are nowhere differentiable.

Additionally, fBm have what is called long range dependence when $H \in (1/2, 1)$. Let us explain this in more detail, first by defining what long range dependence means and then by showing fBm possesses this property. We will say a stationary stochastic process has short range dependence if

$$\int_{\mathbb{R}} |R_X(\tau)| \, d\tau < +\infty$$

and a stationary stochastic process has long range dependence if

$$\int_{\mathbb{R}} |R_X(\tau)|\, d\tau = +\infty \tag{53}$$

Alternate definitions remove the absolute value, so that short range dependence means

$$\int_{\mathbb{R}} R_X(\tau)\, d\tau < +\infty$$

and long range dependence means

$$\int_{\mathbb{R}} R_X(\tau)\, d\tau = +\infty$$

Either way, recall if a stationary stochastic process is centered, i.e., $\mathbb{E}[X(t)] = 0$ for all $t \in \mathbb{R}$, then

$$\forall\, t \in \mathbb{R}\,, \quad R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)]$$

Thus $R_X(\tau)$ measures the correlation between $X(t)$ and $X(t+\tau)$, which has time lag of τ. For stationary processes with short range dependence, this sum total (integration) of this correlation over all possible lags is finite, indicating these correlations must decay rapidly as the lag τ increases. On the other hand, stationary processes with long range dependence have correlations that persist even through large time lags, as indicated by (53). This behavior implies the process has "memory," which can be useful in many modeling situations.

Alternatively, one can say a stationary stochastic process has long range dependence if there exists a real number $\gamma \in (0, 1)$ such that

$$\lim_{\tau \to +\infty} \tau^{\gamma} R_X(\tau) = c_X$$

for some constant $c_X > 0$. This is a characterization of long range dependence in the time domain, and it implies (53). We can also define long range dependence in the frequency domain. Indeed, from the frequency perspective, we say X has long range dependence if there exists a real number $\beta \in (0, 1)$ and a constant $\tilde{c}_X > 0$ such that

$$\lim_{\omega \to 0} |\omega|^{\beta} \widehat{R}_X(\omega) = \tilde{c}_X \tag{54}$$

This frequency condition (54) also implies (53).

Notice the Ornstein-Uhlenbeck process has short range dependence since $R_X(\tau) = e^{-\theta|\tau|}$ and thus

$$\int_{\mathbb{R}} R_X(\tau)\, d\tau = \frac{2}{\theta}$$

or, from the time perspective,

$$\lim_{\tau \to +\infty} \tau^{\gamma} e^{-\theta\tau} = 0\,, \quad \forall\, \gamma \in (0, 1)$$

Fractional Brownian motion for Hurst parameter $H \in (1/2, 1)$ is said to have long range dependence, but an fBm B_H is not stationary, so we need to make sense of this statement. One way to do so is to define a new random process based on the increments of B_H,

$$\widetilde{B}_H = (B_H(t+1) - B_H(t))_{t\in\mathbb{R}}$$

We will not take this path. Another path is to filter a stochastic process that has stationary increments with a wavelet transform. It turns out that the resulting process is stationary. As we mentioned earlier, since fBm is also self-similar, we will also be able to leverage the wavelet coefficients to characterize the self-similarity / regularity of B_H. First let us prove the following proposition.

Proposition 5.25. *Let X be a stochastic process with stationary increments and continuous sample paths. Let ψ be a continuous real valued wavelet with compact support. Then $X * \psi$ is a stationary process.*

Proof. Let $s, t \in \mathbb{R}$. Since ψ has zero average, we have:

$$\begin{aligned}
X * \psi(t) &= \int_{\mathbb{R}} X(t-u)\psi(u)\,du \\
&= \int_{\mathbb{R}} X(t-u)\psi(u)\,du - X(t)\int_{\mathbb{R}} \psi(u)\,du \\
&= \int_{\mathbb{R}} [X(t-u) - X(t)]\psi(u)\,du \\
&\overset{d}{=} \int_{\mathbb{R}} [X(s-u) - X(s)]\psi(u)\,du \\
&= X * \psi(s)
\end{aligned}$$

Letting $s = t + u$ we can apply the same argument to conclude:

$$(X * \psi(t+u))_{t\in\mathbb{R}} \overset{d}{=} (X * \psi(t))_{t\in\mathbb{R}}$$

□

Thus, in particular, $B_H * \psi_s$ is a stationary process for any scale parameter $s > 0$, since ψ_s is a wavelet. This means that its covariance function can be written as

$$\mathrm{Cov}_{B_H*\psi_s}(t, t+\tau) = R_{B_H*\psi_s}(\tau)$$

On the other hand, we cannot directly apply Theorem 5.21 because B_H is not stationary. Nevertheless, we have the following result:

Theorem 5.26. *Let B_H be a fractional Brownian motion with Hurst parameter H. Let ψ be continuous, real valued wavelet with compact support. Then $B_H * \psi$ is a stationary Gaussian process and*

$$\widehat{R}_{B_H*\psi}(\omega) = \frac{\lambda_H}{2|\omega|^{2H+1}}|\widehat{\psi}(\omega)|^2 \tag{55}$$

for some constant $\lambda_H > 0$.

Since $R_{B_H * \psi}$ is not integrable we must understand (55) in the sense of distributions. This means the proof must leverage the distributional definition of the Fourier transform. We give a brief overview now.

Recall the space of Schwartz class functions $\mathcal{S} = \mathcal{S}(\mathbb{R})$, which we originally defined in the proof of Theorem 2.18. The definition was:

$$\mathcal{S} = \left\{ \varphi \in \mathbf{C}^\infty(\mathbb{R}) : \forall\, m, n \in \mathbb{Z} \text{ with } m, n \geq 0\,,\ \sup_{t \in \mathbb{R}} |t|^m |\varphi^{(n)}(t)| < \infty \right\}$$

We note that if $\varphi \in \mathcal{S}$ then $\varphi^{(n)}$ has fast decay, that is

$$|\varphi^{(n)}(t)| \leq \frac{C_{m,n}}{1 + |t|^m}\,, \quad \forall\, m, n \geq 0$$

Now define the dual space of $\mathcal{S}$. It is denoted as $\mathcal{S}'$, and is referred to as the space of *tempered distributions*. It consists of all continuous linear functionals defined on $\mathcal{S}$:

$$\mathcal{S}' = \{T : \mathcal{S} \to \mathbb{C} : T \text{ is continuous and linear}\}$$

In order to understand what T "continuous" means, we need to place a metric on $\mathcal{S}$. To that end, define

$$\|\varphi\|_{m,n} = \sup_{t \in \mathbb{R}} |t|^m |\varphi^{(n)}(t)|$$

Each $\| \cdot \|_{m,n}$ defines a semi-norm on $\mathcal{S}$. We define the metric on $\mathcal{S}$ as

$$d(\varphi_1, \varphi_2) = \sum_{m,n \geq 0} \frac{1}{2^{m+n}} \cdot \frac{\|\varphi_1 - \varphi_2\|_{m,n}}{1 + \|\varphi_1 - \varphi_2\|_{m,n}}$$

Once can prove that $\mathcal{S}$ is complete with the metric $d(\varphi_1, \varphi_2)$. Furthermore, if $T \in \mathcal{S}'$ then there exists some $d \in \mathbb{Z}$ and constants $c_{m,n} \geq 0$ such that

$$|T(\varphi)| \leq \sum_{m=0}^{d} \sum_{n=0}^{d} c_{m,n} \|\varphi\|_{m,n} \tag{56}$$

Conversely, if T is a linear functional and (56) holds for some $d \in \mathbb{Z}$ and $c_{m,n} \geq 0$, then T is continuous and $T \in \mathcal{S}'$.

Now let us give some examples of tempered distributions $T \in \mathcal{S}'$.

Example 5.27. The Dirac distribution $\delta : \mathcal{S} \to \mathbb{C}$, defined as:

$$\delta(\varphi) = \varphi(0)$$

We can generalize it to $\delta_t : \mathcal{S} \to \mathbb{C}$,

$$\delta_t(\varphi) = \varphi(t)$$

Example 5.28. Let f be Lebesgue measurable and

$$|f(t)| \leq g(t)(1 + |t|^m)$$

for some $m \geq 0$ with $g \in \mathbf{L}^1(\mathbb{R})$ and $g(t) \geq 0$. Then $T_f \in \mathcal{S}'$ where

$$T_f(\varphi) = \int_{\mathbb{R}} f(t)\varphi(t)\, dt$$

Indeed,

$$\begin{aligned} |T_f(\varphi)| &\leq \int_{\mathbb{R}} g(t)(1 + |t|^m)|\varphi(t)|\, dt \\ &\leq \sup_{u \in \mathbb{R}} (1 + |u|^m)|\varphi(u)| \cdot \int_{\mathbb{R}} g(t)\, dt \\ &< \infty \end{aligned}$$

Note that f does not have to be in $\mathbf{L}^1(\mathbb{R})$ or $\mathbf{L}^2(\mathbb{R})$.

Now we want to define the Fourier transform of a tempered distribution $T \in \mathcal{S}'$, which we will denote by $\widehat{T}$. We first note that for $\varphi \in \mathcal{S}$, we can use the $\mathbf{L}^1(\mathbb{R})$ definition of the Fourier transform to define $\widehat{\varphi}$:

$$\widehat{\varphi}(\omega) = \int_{\mathbb{R}} \varphi(t) e^{-i\omega t}\, dt$$

Since $\varphi \in \mathbf{C}^\infty(\mathbb{R})$ and $\varphi^{(n)}(t)$ has fast decay for each $n \geq 0$, $\widehat{\varphi} \in \mathbf{C}^\infty(\mathbb{R})$ and $\widehat{\varphi}^{(n)}(\omega)$ has fast decay for each $n \geq 0$ as well. Therefore $\widehat{\varphi} \in \mathcal{S}$. Furthermore, for $f, \varphi \in \mathcal{S}$, using Fubini's Theorem we have

$$\begin{aligned} T_{\widehat{f}}(\varphi) = \int_{\mathbb{R}} \widehat{f}(t)\varphi(t)\, dt &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(\omega) e^{-i\omega t}\, d\omega \right] \varphi(t)\, dt \\ &= \int_{\mathbb{R}} f(\omega) \left[\int_{\mathbb{R}} \varphi(t) e^{-i\omega t}\, dt \right] d\omega \\ &= \int_{\mathbb{R}} f(\omega)\widehat{\varphi}(\omega)\, d\omega \\ &= T_f(\widehat{\varphi}) \end{aligned}$$

Inspired by this correspondence we make the following definition.

Definition 5.29. The Fourier transform of a tempered distribution $T \in \mathcal{S}'$ is the tempered distribution $\widehat{T} \in \mathcal{S}'$ defined as

$$\widehat{T}(\varphi) := T(\widehat{\varphi}), \quad \forall\, \varphi \in \mathcal{S}$$

Example 5.30. For the Dirac distribution

$$\widehat{\delta}(\varphi) = \delta(\widehat{\varphi}) = \widehat{\varphi}(0) = \int_{\mathbb{R}} \varphi(t)\,dt = T_{\chi_{\mathbb{R}}}(\varphi)$$

where $\chi_A(t) = 1$ if $t \in A \subseteq \mathbb{R}$ and $\chi_A(t) = 0$ if $t \notin A$. Hence the interpretation from earlier in the course that $\widehat{\delta}(\omega) = 1$ for all $\omega \in \mathbb{R}$.

Example 5.31. Let $f \in \mathbf{L}^1(\mathbb{R})$. Then using Fubini's Theorem:

$$\begin{aligned}
\widehat{T}_f(\varphi) = T_f(\widehat{\varphi}) &= \int_{\mathbb{R}} f(t)\widehat{\varphi}(t)\,dt \\
&= \int_{\mathbb{R}} f(t)\left[\int_{\mathbb{R}} \varphi(\omega)e^{-i\omega t}\,d\omega\right]dt \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(t)e^{-i\omega t}\,dt\right]\varphi(\omega)\,d\omega \\
&= \int_{\mathbb{R}} \widehat{f}(\omega)\varphi(\omega)\,d\omega = T_{\widehat{f}}(\varphi)
\end{aligned}$$

and thus the $\mathbf{L}^1(\mathbb{R})$ definition of the Fourier transform agrees with the distributional definition of the Fourier transform.

Our last example is more complicated, and needed for the proof of Theorem 5.26, so we collect it in the following lemma.

Lemma 5.32. *Let $f(t) = |t|^\alpha$ for $\alpha > 0$. Then, in the sense of distributions,*

$$\widehat{f}(\omega) = \lambda_\alpha |\omega|^{-(1+\alpha)}$$

That is

$$\widehat{T}_f(\varphi) = \lambda_\alpha \cdot \text{p.v.} \int_{\mathbb{R}} |\omega|^{-(1+\alpha)}\varphi(\omega)\,d\omega = \lim_{\epsilon \to 0^+} \int_{|\omega|>\epsilon} |\omega|^{-(1+\alpha)}\varphi(\omega)\,d\omega$$

Proof. For the purposes of this proof define $\varphi_s(t) = \varphi(st)$. A tempered distribution $T \in \mathcal{S}'$ is homogeneous of order α if

$$\forall\, s > 0\,,\; \varphi \in \mathcal{S}\,, \quad T(\varphi) = s^{1+\alpha}T(\varphi_s)$$

We first show that if T is homogeneous of order α, then $\widehat{T}$ is homogeneous of order $-(1+\alpha)$. Indeed we know:

$$\widehat{\varphi_s}(\omega) = s^{-1}\widehat{\varphi}(s^{-1}\omega) = s^{-1}\widehat{\varphi}_{s^{-1}}(\omega)$$

Therefore:

$$\begin{aligned}
\widehat{T}(\varphi_s) &= T(\widehat{\varphi_s}) \\
&= s^{-1}T(\widehat{\varphi}_{s^{-1}}) \\
&= s^{-1}s^{1+\alpha}T(\widehat{\varphi}) \\
&= s^{\alpha}\widehat{T}(\varphi)
\end{aligned}$$

Rearranging:

$$\widehat{T}(\varphi) = s^{-\alpha}\widehat{T}(\varphi_s) = s^{1-(1+\alpha)}\widehat{T}(\varphi)$$

Now observe that T_f with $f(t) = |t|^\alpha$ is homogeneous of order α since

$$T_f(\varphi_s) = \int_{\mathbb{R}} |t|^\alpha \varphi(st)\, dt = \int_{\mathbb{R}} |u/s|^\alpha\, \varphi(u)\frac{du}{s} = s^{-(1+\alpha)} \int_{\mathbb{R}} |u|^\alpha \varphi(u)\, du s^{-(1+\alpha)} T_f(\varphi)$$

Therefore $\widehat{T}_f$ must be homogeneous of order $-(1+\alpha)$. Additionally, $f(t) = |t|^\alpha$ is even, which means $T_f(\varphi)$ is "even," where the latter means

$$T_f(\varphi_{-1}) = T_f(\varphi)$$

Furthermore, since $f(t) = |t|^\alpha$ is real valued, $T_f(\varphi)$ is real valued for all real valued $\varphi \in \mathcal{S}$. It follows that $\widehat{T}(\varphi)$ must also be even and real valued if φ is real valued. But the only distributions which are homogeneous of order $-(1+\alpha)$, even, and real valued, are $c|\omega|^{-(1+\alpha)}$. □

Proof of Theorem 5.26. Set $f(t) = R_{B_H * \psi}(t)$. We compute:

$$\begin{aligned}
\widehat{T}_f &= T_f(\widehat{\varphi}) \\
&= \int_{\mathbb{R}} R_{B_H*\psi}(t)\widehat{\varphi}(t)\, dt \\
&= \int_{\mathbb{R}} \mathbb{E}[B_H * \psi(0) B_H * \psi(t)]\widehat{\varphi}(t)\, dt \\
&= \int_{\mathbb{R}} \mathbb{E}\left[\int_{\mathbb{R}} B_H(u)\psi(-u)\, du \cdot \int_{\mathbb{R}} B_H(v)\psi(t-v)\, dv\right]\widehat{\varphi}(t)\, dt \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}[B_H(u)B_H(v)]\psi(-u)\psi(t-v)\, du\, dv\right]\widehat{\varphi}(t)\, dt \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} (|u|^{2H} + |v|^{2H} - |u-v|^{2H})\psi(-u)\psi(t-v)\, du\, dv\right]\widehat{\varphi}(t)\, dt \\
&= \frac{1}{2}\int_{\mathbb{R}} \Bigg[\int_{\mathbb{R}} |u|^{2H}\psi(-u) \underbrace{\int_{\mathbb{R}} \psi(t-v)\, dv}_{=0}\, du + \int_{\mathbb{R}} |v|^{2H}\psi(t-v) \underbrace{\int_{\mathbb{R}} \psi(-u)\, du}_{=0}\, dv - \cdots \\
&\qquad \cdots - \int_{\mathbb{R}}\int_{\mathbb{R}} |u-v|^{2H}\psi(-u)\psi(t-v)\, du\, dv\Bigg]\widehat{\varphi}(t)\, dt \\
&= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |u-v|^{2H}\psi(-u)\psi(t-v)\widehat{\varphi}(t)\, du\, dv\, dt \qquad \text{(CoV: } x = t - v\text{)} \\
&= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |t-(u+x)|^{2H}\psi(-u)\psi(x)\widehat{\varphi}(t)\, du\, dx\, dt \\
&= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi(-u)\psi(x) \underbrace{\left[\int_{\mathbb{R}} |t-(u+x)|^{2H}\widehat{\varphi}(t)\, dt\right]}_{I} du\, dx
\end{aligned} \tag{57}$$

Now let us evaluate the integral I. First make a change of variables $y = t - (u+x)$. One obtains:

$$\begin{aligned} I &= \int_{\mathbb{R}} |y|^{2H} \widehat{\varphi}(y+u+x)\, dy \\ &= \int_{\mathbb{R}} |y|^{2H} \left[\int_{\mathbb{R}} \varphi(\omega) e^{-i\omega(y+u+x)}\, d\omega \right] dy \\ &= \int_{\mathbb{R}} |y|^{2H} \left[\int_{\mathbb{R}} e^{i\omega(u+x)} \varphi(\omega) e^{-i\omega y}\, d\omega \right] dy \\ &= \int_{\mathbb{R}} |y|^{2H} \mathcal{F}(M_{-(u+x)}\varphi)(y)\, dy \qquad [(M_v \varphi)(y) = e^{ivy}\varphi(y)] \\ &= -\int_{\mathbb{R}} \lambda_H |\omega|^{-(2H+1)} e^{-i\omega(u+x)} \varphi(\omega)\, d\omega \end{aligned} \tag{58}$$

where in the last line we used Lemma 5.32. Now plug (58) into (57) to obtain:

$$\begin{aligned} (57) &= \frac{\lambda_H}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(-u)\psi(x) \left[\int_{\mathbb{R}} |\omega|^{-(2H+1)} e^{-i\omega(u+x)} \varphi(\omega)\, d\omega \right] du\, dx \\ &= \frac{\lambda_H}{2} \int_{\mathbb{R}} |\omega|^{-(2H+1)} \varphi(\omega) \int_{\mathbb{R}} \psi(-u) e^{-i\omega u}\, du \int_{\mathbb{R}} \psi(x) e^{-i\omega x}\, dx\, d\omega \\ &= \int_{\mathbb{R}} \frac{\lambda_H}{2} |\omega|^{-(2H+1)} \widehat{\psi}^*(\omega) \widehat{\psi}(\omega) \varphi(\omega)\, d\omega \\ &= \int_{\mathbb{R}} \frac{\lambda_H}{2} |\omega|^{-(2H+1)} |\widehat{\psi}(\omega)|^2 \varphi(\omega)\, d\omega \end{aligned}$$

We conclude that, in the distributional sense,

$$\widehat{R}_{B_H * \psi}(\omega) = \frac{\lambda_H}{2|\omega|^{2H+1}} |\widehat{\psi}(\omega)|^2$$

□

It is tempting to think of the "power spectral density of B_H" as $(\lambda_H/2)|\omega|^{-(2H+1)}$ but this is not quite correct, and would lead, for example, to the wrong interpretation of its long range dependence property. Recall that since ψ is a wavelet,

$$\widehat{\psi}(\omega) = O(\omega) \text{ as } \omega \to 0$$

and thus $|\widehat{\psi}(\omega)|^2 = O(|\omega|^2)$. It follows that

$$\widehat{R}_{B_H * \psi}(\omega) = O(|\omega|^{1-2H})$$

Thus when $H \in (1/2, 1)$ we see that

$$\lim_{\omega \to 0} |\omega|^{2H-1} \widehat{R}_{B_H * \psi}(\omega) = c > 0$$

with $0 < 2H - 1 < 1$. Therefore we see that $B_H * \psi$ has long range dependence. Notice, however, for $H \in (0, 1/2)$ the same cannot be said since

$$\forall\, H \in (0, 1/2)\,, \quad \lim_{\omega \to 0} \widehat{R}_{B_H * \psi}(\omega) = c \cdot \lim_{\omega \to 0} |\omega|^{1-2H} = 0$$

Using the self-similarity of fBm, one can also show:

$$WB_H(u, s) \overset{d}{=} s^{H+1/2} WB_H\left(\frac{u}{s}, 1\right)$$

We leave the details as an exercise.

Exercise 54. Read Section 6.4 of *A Wavelet Tour of Signal Processing*.

Exercise 55. Let ψ be a continuous, compactly supported, real valued wavelet. Recall $Wf(u, s) = f * \overline{\psi}_s(u)$ with $\overline{\psi}(t) = \psi(-t)$. Prove:

$$\mathbb{E}[WB_H(u, s) WB_H(v, s)] = -\frac{s^{2H+1}}{2} \int_{\mathbb{R}} |t|^{2H} \psi * \overline{\psi}\left(\frac{u - v}{s} - t\right) dt$$

Observe that since B_H is a Gaussian process and

$$\mathbb{E}[WB_H(u, s)] = \mathbb{E}\left[\int_{\mathbb{R}} B_H(t) \psi_s(t - u)\, dt\right] = \int_{\mathbb{R}} \mathbb{E}[B_H(t)] \psi_s(t - u)\, dt = 0$$

you now have a direct proof that $(WB_H(u, s))_{u \in \mathbb{R}}$ is a stationary stochastic process.

Exercise 56. Let X be a second order stochastic process that is self-similar of order H with continuous sample paths, and let ψ be a continuous, compactly supported real valued wavelet.

(a) Prove:

$$X * \psi_s(u) \overset{d}{=} s^{H+1/2} X * \psi\left(\frac{u}{s}\right)$$

Conclude that if X also has stationary increments then:

$$\mathbb{E}[|X * \psi_s(u)|] = s^{H+1/2} \mathbb{E}[|X * \psi(0)|]$$

(b) Suppose X also has stationary increments. Prove:

$$\frac{\mathbb{E}[||X * \psi_{s_1}| * \psi_{s_2}(u)|]}{s_1^{1/2} \mathbb{E}[|X * \psi_{s_1}(u)|]} = \frac{\mathbb{E}[||X * \psi| * \psi_{s_2/s_1}(0)|]}{\mathbb{E}[|X * \psi(0)|]}$$

The numerator of the left hand side is called a wavelet scattering moment. Give an interpretation of this result.

Exercise 57. One can obtain realizations of fractional Brownian motion in MATLAB using the `wfbm` function (`https://www.mathworks.com/help/wavelet/ref/wfbm.html`) or in Python using the `fbm` package (available at: `https://pypi.org/project/fbm/`).

(a) Generate realizations of fractional Brownian motion for three different Hurst parameters H, one with $H < 1/2$, one with $H = 1/2$ (regular Brownian motion), and one with $H > 1/2$. Provide a plot of each realization. Using your code for the real valued wavelet transform from Exercise 44, compute the wavelet transform for each realization and plot the wavelet coefficients as in Figure 6.22(b) from the book.

Remark: You should generate long realizations of fBm with $N \geq 10000$.

(b) Now estimate the Hurst parameter H using the moments computed in Exercise 56(a). Do so by noting that Exercise 56(a) implies

$$F(\log_2 s) := \log_2 \mathbb{E}[|B_H * \psi_s(u)|] = (H + 1/2) \log_2 s + \log_2 \mathbb{E}[|B_H * \psi(0)|] \tag{59}$$

Since $|B_H * \psi_s|$ is stationary for each $s > 0$, the function $F(\log_2 s)$ on the left hand side of (59) does not depend on u and can be considered as a function of $\log_2 s$. The right hand side of (59) shows $F(\log_2 s)$ is linear with a slope of $H + 1/2$. Plot $F(\log_2 s) = \log_2 \mathbb{E}[|B_H * \psi_s(u)|]$ as a function of $\log_2 s$ for your three realizations from part (a). Estimate the slope numerically and compare it to the true value of H.

Remark: To estimate $\mathbb{E}[|B_H * \psi_s(u)|]$ *note that* $|B_H * \psi_s|$ *is stationary. For a stationary process* Y*, one can estimate* $\mathbb{E}[Y(t)] = \mathbb{E}[Y(0)]$ *by computing:*

$$\frac{1}{N} \sum_{i=1}^{N} Y(t_i) \approx \mathbb{E}[Y(0)]$$

for large N.

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