Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 23 & 24: Introduction to Frames

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Lecturer: Matthew Hirn

6 Frames

Chapter 5 of A Wavelet Tour of Signal Processing.

The windowed Fourier transform $Sf(u,\xi)$ and the wavelet transform $Wf(u,s)$ are examples of signal analysis operators, which can be put in a more general context via Frame theory. Frame theory will give us the mathematical foundation to consider general dictionaries of time frequency atoms. It will, additionally, give as the mathematical framework to synthesize signals, not just analyze them. This will be useful for, amongst other reasons, obtaining sparse compression of signals using just their wavelet modulus maxima coefficients. For now we leave wavelets to study frames, but we will return to wavelets possessing the framework to not only complete their story, but also the tools to chart a path forward into signal analysis via more general dictionaries.

6.1 Frames and Riesz Bases

Section 5.1 of A Wavelet Tour of Signal Processing.

6.1.1 Stable Analysis and Synthesis Operators

Section 5.1.1 of A Wavelet Tour of Signal Processing.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|f\| = \sqrt{\langle f, f \rangle}$. The main examples we will want to keep in the back of our mind are the ones we have encountered thus far in the course, i.e., $\mathbf{L}^2(\mathbb{R})$, $\boldsymbol{\ell}^2$, and $\mathbb{R}^N$ or $\mathbb{C}^N$. Consider a dictionary

$$\mathcal{D} = \{\phi_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{H}$$

consisting of atoms $\phi_\gamma \in \mathcal{H}$, in which the index set Γ is either finite or countable. The *analysis operator* associated to $\mathcal{D}$ is:

$$\Phi f(\gamma) = \langle f, \phi_\gamma \rangle, \quad \gamma \in \Gamma, \ \ f \in \mathcal{H}$$

The dictionary $\mathcal{D}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{\gamma \in \Gamma} |\langle f, \phi_\gamma \rangle|^2 \leq B\|f\|^2 \tag{60}$$

When $A = B$ the frame is *tight*. If the atoms in $\mathcal{D}$ are independent, then the frame is not redundant and it is called a *Riesz basis*. We shall see later that frames define invertible operators on image(Φ).

We remark that if a Hilbert space $\mathcal{H}$ admits a frame $\mathcal{D}$, then $\mathcal{H}$ must be separable. Indeed, suppose that $\langle f, \phi_\gamma \rangle = 0$ for all $\gamma \in \Gamma$. Then using the lower bound of (60), we obtain:

$$A\|f\|^2 \leq \sum_{\gamma \in \Gamma} |\langle f, \phi_\gamma \rangle|^2 = 0 \Longrightarrow f = 0$$

Thus the only element of $\mathcal{H}$ orthogonal to every $\phi_\gamma \in \mathcal{D}$ is $f = 0$. It follows (with some work) that $\mathcal{D}$ must be a complete set of functions in $\mathcal{H}$. This means that for each $f \in \mathcal{H}$ and for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, $\{\gamma_n\}_{n=1}^N \subset \Gamma$ and coefficients $\{c_n\}_{n=1}^N \subset \mathbb{C}$ such that

$$\left\| f - \sum_{n=1}^{N} c_n \phi_{\gamma_n} \right\| \leq \varepsilon$$

Since we can additionally take the coefficients $\{c_n\}_{n=1}^N$ to have rational real and imaginary parts, we have found a dense subset of $\mathcal{H}$.

The analysis operator Φ analyzes a signal $f \in \mathcal{H}$ by testing it against the dictionary atoms ϕ_γ. The adjoint of Φ defines a synthesis operator, which we now explain. Consider the space of $\boldsymbol{\ell}^2$ sequences indexed by Γ:

$$\boldsymbol{\ell}^2(\Gamma) = \{a : \|a\|^2 = \sum_{\gamma \in \Gamma} |a[\gamma]|^2 < \infty\}$$

Notice that the frame condition (60) guarantees that

$$\Phi : \mathcal{H} \to \boldsymbol{\ell}^2(\Gamma)$$

Therefore Φ has an adjoint

$$\Phi^* : \boldsymbol{\ell}^2(\Gamma) \to \mathcal{H}$$

which is defined through the following relation:

$$\langle \Phi^* a, f \rangle_{\mathcal{H}} = \langle a, \Phi f \rangle_{\boldsymbol{\ell}^2(\Gamma)}$$

where the subscript on the inner products $\langle \cdot, \cdot \rangle$ is written to emphasize the space over which the inner product is computed (moving forward we will drop this subscript and infer the space from the context). Notice that

$$\begin{aligned}\langle a, \Phi f\rangle &= \sum_{\gamma\in\Gamma} a[\gamma]\langle f, \phi_\gamma\rangle^* \\ &= \sum_{\gamma\in\Gamma} a[\gamma]\langle \phi_\gamma, f\rangle \\ &= \sum_{\gamma\in\Gamma}\langle a[\gamma]\phi_\gamma, f\rangle \\ &= \left\langle \sum_{\gamma\in\Gamma} a[\gamma]\phi_\gamma, f\right\rangle\end{aligned}$$

from which it follows that

$$\Phi^* a = \sum_{\gamma} a[\gamma]\phi_\gamma$$

We refer to Φ^* as the *synthesis* operator since it synthesizes signals in $\mathcal{H}$ from the sequence $a \in \boldsymbol{\ell}^2(\Gamma)$.

Notice that the frame condition (60) can be rewritten as:

$$A\|f\|^2 \leq \|\Phi f\|^2 = \langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2$$

where

$$\Phi^*\Phi f = \sum_{\gamma\in\Gamma}\langle f, \phi_\gamma\rangle\phi_\gamma \tag{61}$$

Notice that (61) looks exactly like the formula you get for expanding a vector f in an orthonormal basis. However, Φ here is a frame and so in general (61) will not return f but rather another element of $\mathcal{H}$. Back to the point at hand, it follows that we can take A and B as:

$$\begin{aligned}A &= \inf_{f\in\mathcal{H}} \frac{\langle \Phi^*\Phi f, f\rangle}{\|f\|^2} \\ B &= \sup_{f\in\mathcal{H}} \frac{\langle \Phi^*\Phi f, f\rangle}{\|f\|^2}\end{aligned}$$

This is just the infimum and supremum of the Rayleigh quotient of $\Phi^*\Phi$. In finite dimensions, this implies that A is the smallest eigenvalue of $\Phi^*\Phi$ and B is the largest eigenvalue of $\Phi^*\Phi$; note that the eigenvalues of $\Phi^*\Phi$ are the singular values of Φ. The next theorem shows that if the frame analysis operator is stable (as defined by the frame condition (60)), then the frame synthesis operator obeys a similar stability condition.

Theorem 6.1. *A dictionary $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ is a frame with bounds $0 < A \leq B < \infty$ if and only if*

$$A\|a\|^2 \leq \left\|\sum_{\gamma\in\Gamma} a[\gamma]\phi_\gamma\right\|^2 \leq B\|a\|^2, \quad \forall\, a \in \text{image}(\Phi)$$

Proof. Note that

$$\left\| \sum_{\gamma \in \Gamma} a[\gamma] \phi_\gamma \right\|^2 = \langle \Phi^* a, \Phi^* a \rangle = \langle \Phi \Phi^* a, a \rangle$$

The theorem will thus follow if we can show that

$$\inf_{f \in \mathcal{H}} \frac{\langle \Phi^* \Phi f, f \rangle}{\|f\|^2} = \inf_{a \in \text{image}(\Phi)} \frac{\langle \Phi \Phi^* a, a \rangle}{\|a\|^2} \tag{62}$$

and

$$\sup_{f \in \mathcal{H}} \frac{\langle \Phi^* \Phi f, f \rangle}{\|f\|^2} = \sup_{a \in \text{image}(\Phi)} \frac{\langle \Phi \Phi^* a, a \rangle}{\|a\|^2} \tag{63}$$

Let us first consider the case of a finite dimensional Hilbert space. In this case $\mathcal{H} \cong \mathbb{R}^N$ or $\mathcal{H} \cong \mathbb{C}^N$. Suppose that $\mathcal{D}$ is a frame and let λ be an eigenvalue of $\Phi^*\Phi$ with eigenvector $f_\lambda \neq 0$. Note the frame condition implies $\Phi^*\Phi$ is invertible and every eigenvalue satisfies $A \leq \lambda \leq B$. Furthermore $\Phi^*\Phi$ can be identified with an $N \times N$ matrix. We claim that $\Phi f_\lambda \in \text{image}(\Phi)$ is an eigenvector of $\Phi\Phi^*$ also with eigenvalue λ; indeed:

$$\Phi \Phi^* (\Phi f_\lambda) = \Phi \Phi^* \Phi f_\lambda = \lambda \Phi f_\lambda$$

Furthermore $\Phi f_\lambda \neq 0$ since the frame bounds (60) imply that $\|\Phi f_\lambda\|^2 \geq A \|f_\lambda\|^2$. Since $\dim(\text{image}(\Phi)) = N$, we have shown the eigenvalues of $\Phi^*\Phi$ and $\Phi\Phi^*|_{\text{image}(\Phi)}$ are identical and we conclude that (62) and (63) hold.

Now suppose that $\mathcal{H}$ is infinite dimensional and $\mathcal{D}$ is a frame for $\mathcal{H}$. From our previous discussion, we know that $\mathcal{H}$ is separable, which means that $\mathcal{H}$ has a countable orthonormal basis. Let $\mathcal{B} = \{e_1, e_2, \ldots\} \subset \mathcal{H}$ be such a basis. Define

$$\mathcal{H}_N = \text{span}\{e_1, \ldots, e_N\} \subset \mathcal{H}$$

Let $\Phi_N = \Phi|_{\mathcal{H}_N}$, that is Φ_N is the restriction of Φ to $\mathcal{H}_N$. Notice that $\lim_{N \to \infty} \mathcal{H}_N = \mathcal{H}$ and $\lim_{N \to \infty} \text{image}(\Phi_N) = \text{image}(\Phi)$. Using the proof for the finite dimensional case, we then have:

$$\begin{aligned}
\inf_{f \in \mathcal{H}} \frac{\langle \Phi^* \Phi f, f \rangle}{\|f\|^2} &= \lim_{N \to \infty} \inf_{f \in \mathcal{H}_N} \frac{\langle \Phi^* \Phi f, f \rangle}{\|f\|^2} \\
&= \lim_{N \to \infty} \inf_{a \in \text{image}(\Phi_N)} \frac{\langle \Phi \Phi^* a, a \rangle}{\|a\|^2} = \inf_{a \in \text{image}(\Phi)} \frac{\langle \Phi \Phi^* a, a \rangle}{\|a\|^2}
\end{aligned}$$

The proof for the supremum is identical.

□

The operator $\Phi\Phi^* : \text{image}(\Phi) \to \text{image}(\Phi)$ is the *Gram "matrix"*. It is defined as:

$$\Phi \Phi^* a[\gamma] = \sum_{m \in \Gamma} a[m] \langle \phi_m, \phi_\gamma \rangle, \quad \forall\, a \in \text{image}(\Phi)$$

The next theorem shows that the redundancy of a finite frame in finite dimensions is easy to measure, and is the obvious answer.

Theorem 6.2. *Let $\mathcal{D} = \{\phi_n\}_{n=1}^P$ be a finite frame for $\mathbb{R}^N$ or $\mathbb{C}^N$ in which $\|\phi_n\| = 1$ for all $1 \leq n \leq P$. Then the frame bounds satisfy:*

$$A \leq \frac{P}{N} \leq B$$

and the frame is tight if and only if $A = B = P/N$.

The proof is on page 157 of *A Wavelet Tour of Signal Processing* and is quite simple. Tight frames are easy to come up with by concatenating orthonormal bases. For $1 \leq k \leq K$, suppose that $\{\phi_{k,\gamma}\}_{\gamma\in\Gamma}$ is an orthonormal basis for $\mathcal{H}$. Since it is an orthonormal basis we have:

$$\sum_{\gamma\in\Gamma} |\langle f, \phi_{k,\gamma}\rangle|^2 = \|f\|^2$$

The dictionary

$$\mathcal{D} = \{\phi_{k,\gamma}\}_{\gamma\in\Gamma,\ 1\leq k\leq K}$$

is a tight frame with $A = B = K$; indeed:

$$\sum_{k=1}^{K}\sum_{\gamma\in\Gamma} |\langle f, \phi_{k,\gamma}\rangle|^2 = \sum_{k=1}^{K} \|f\|^2 = K\|f\|^2$$

Exercise 58. Read Section 5.1.1 of *A Wavelet Tour of Signal Processing.*

6.1.2 Dual Frame and Pseudo Inverse

Section 5.1.2 of A Wavelet Tour of Signal Processing.

If $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ is a frame but not a Riesz basis, then the frame analysis operator Φ admits an infinite number of left inverses M such that

$$M\Phi f = f, \quad \forall\, f \in \mathcal{H}$$

This is because of the redundancy of $\mathcal{D}$, which ensures that $\text{image}(\Phi)^\perp \neq \{0\}$, and so the left inverse is free to map $a \in \text{image}(\Phi)^\perp$ to any function $g \in \mathcal{H}$. The pseudo-inverse, written as $\Phi^\dagger$, is the left inverse M that maps $\text{image}(\Phi)^\perp$ to 0:

$$\Phi^\dagger\Phi f = f,\ \forall\, f \in \mathcal{H} \qquad \text{and} \qquad \Phi^\dagger a = 0,\ \forall\, a \in \text{image}(\Phi)^\perp$$

The next theorem computes the pseudo-inverse explicitly.

Theorem 6.3. *If $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ is a frame then $\Phi^*\Phi$ is invertible and*

$$\Phi^\dagger = (\Phi^*\Phi)^{-1}\Phi^*$$

Proof. First recall that we can rewrite the frame condition (60) as:

$$A\|f\|^2 \leq \langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2$$

Thus

$$\Phi^*\Phi f = 0 \Longleftrightarrow f = 0$$

and so $\Phi^*\Phi$ is invertible. It follows that

$$(\Phi^*\Phi)^{-1}(\Phi^*\Phi)f = f$$

which shows that $M = (\Phi^*\Phi)^{-1}\Phi^*$ is a left inverse for Φ. Now we show that $M = \Phi^\dagger$.

We first show that $\text{null}(\Phi^*) = \text{image}(\Phi)^\perp$. Let $a \in \text{null}(\Phi^*)$ and $b \in \text{image}(\Phi)$ with $\Phi f = b$. Then:

$$\langle a, b\rangle = \langle a, \Phi f\rangle = \langle \Phi^* a, f\rangle = \langle 0, f\rangle = 0$$

Thus $\text{null}(\Phi^*) \subseteq \text{image}(\Phi)^\perp$. Similarly, now let $a \in \text{image}(\Phi)^\perp$, so that:

$$\begin{aligned} a \in \text{image}(\Phi)^\perp &\Longrightarrow \langle a, \Phi f\rangle = 0\,, \quad \forall\, f \in \mathcal{H} \\ &\Longrightarrow \langle \Phi^* a, f\rangle = 0\,, \quad \forall\, f \in \mathcal{H} \\ &\Longrightarrow \Phi^* a = 0 \\ &\Longrightarrow a \in \text{null}(\Phi^*) \end{aligned}$$

Therefore $\text{image}(\Phi)^\perp \subseteq \text{null}(\Phi^*)$ and we conclude that $\text{image}(\Phi)^\perp = \text{null}(\Phi^*)$. But then

$$(\Phi^*\Phi)^{-1}\Phi^* a = 0, \quad \forall\, a \in \text{image}(\Phi)^\perp = \text{null}(\Phi^*)$$

and so we have $\Phi^\dagger = (\Phi^*\Phi)^{-1}\Phi^*$. □

The pseudo-inverse implements a signal synthesis with the *(canonical) dual frame*, defined by:

$$\widetilde{\phi}_\gamma = (\Phi^*\Phi)^{-1}\phi_\gamma$$

which has associated frame analysis operator

$$\widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma\rangle$$

The next theorem shows that the dual frame synthesis operator is indeed the pseudo-inverse of the original frame analysis operator, and that the dual frame is in fact a frame.

Theorem 6.4. *Let $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ be a frame with frame bounds $0 < A \leq B < \infty$. Then the dual frame synthesis operator satisfies*

$$\widetilde{\Phi}^* = \Phi^\dagger \tag{64}$$

and thus

$$f = \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma\rangle \widetilde{\phi}_\gamma = \sum_{\gamma\in\Gamma} \langle f, \widetilde{\phi}_\gamma\rangle \phi_\gamma \tag{65}$$

Furthermore, the dual dictionary

$$\widetilde{\mathcal{D}} = \{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$$

is a frame (hence the name dual frame) with frame bounds $0 < 1/B \leq 1/A < \infty$*, meaning that*

$$\frac{1}{B}\|f\|^2 \leq \sum_{\gamma\in\Gamma} |\langle f, \widetilde{\phi}_\gamma\rangle|^2 \leq \frac{1}{A}\|f\|^2, \quad \forall\, f \in \mathcal{H} \tag{66}$$

If the frame is tight (i.e., $A = B$*), then*

$$\widetilde{\phi}_\gamma = \frac{1}{A}\phi_\gamma$$

To prove this theorem, we will need the following lemma.

Lemma 6.5. *If* $L : \mathcal{H} \to \mathcal{H}$ *is a self-adjoint operator such that there exists* $0 < A \leq B < \infty$ *satisfying*

$$A\|f\|^2 \leq \langle Lf, f\rangle \leq B\|f\|^2, \quad \forall\, f \in \mathcal{H} \tag{67}$$

then L *is invertible and*

$$\frac{1}{B}\|f\|^2 \leq \langle L^{-1}f, f\rangle \leq \frac{1}{A}\|f\|^2, \quad \forall\, f \in \mathcal{H} \tag{68}$$

Proof. Suppose first that $\mathcal{H}$ is finite dimensional of dimension N. Since L is self-adjoint, it has an orthonormal set of eigenvectors $e_1, \ldots, e_N \in \mathcal{H}$ with eigenvalues $\lambda_1, \ldots, \lambda_N$ such that

$$Le_k = \lambda_k e_k, \quad \forall\, 1 \leq K \leq N$$

Equation (67) implies that $A \leq \lambda_k \leq B$ for each k. The operator L is therefore invertible, and it's eigenvalues are λ_k^{-1} with the same orthonormal eigenvectors e_k for $1 \leq k \leq N$. It follows that (68) must hold. The proof is extended to infinite dimensions using the same technique as in the proof of Theorem 6.1. □

Proof of Theorem 6.4. We first rewrite the dual analysis operator (noting that $\Phi^*\Phi$ is self-adjoint, and thus so is $(\Phi^*\Phi)^{-1}$):

$$\begin{aligned}\widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma\rangle &= \langle f, (\Phi^*\Phi)^{-1}\phi_\gamma\rangle \\ &= \langle (\Phi^*\Phi)^{-1} f, \phi_\gamma\rangle \\ &= \Phi(\Phi^*\Phi)^{-1} f(\gamma)\end{aligned}$$

Thus

$$\widetilde{\Phi} = \Phi(\Phi^*\Phi)^{-1}$$

and we compute:

$$\widetilde{\Phi}^* = (\Phi^*\Phi)^{-1}\Phi^* = \Phi^\dagger$$

That proves (64).

Note that (65) can be written as:

$$I = \widetilde{\Phi}^*\Phi = \Phi^*\widetilde{\Phi}$$

where I is the identity operator. Since $\widetilde{\Phi}^* = \Phi^\dagger$, we have

$$\widetilde{\Phi}^*\Phi = \Phi^\dagger\Phi = I \tag{69}$$

Using the facts that $(\widetilde{\Phi}^*\Phi)^* = \Phi^*\widetilde{\Phi}$ and $I^* = I$, and taking the adjoint of both sides of (69), we obtain the second equality.

For the proof of (66), we use Lemma 6.5. Recall that the frame conditions can be rewritten as:

$$A\|f\|^2 \le \langle \Phi^*\Phi f, f\rangle \le B\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

Applying Lemma 6.5 to $L = \Phi^*\Phi$ proves that

$$\frac{1}{B}\|f\|^2 \le \langle (\Phi^*\Phi)^{-1} f, f\rangle \le \frac{1}{A}\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

Furthermore, using the first part of the proof we have:

$$\begin{aligned}
\sum_{\gamma\in\Gamma} |\langle f, \widetilde{\phi}_\gamma\rangle|^2 &= \|\widetilde{\Phi} f\|^2 \\
&= \langle \Phi(\Phi^*\Phi)^{-1} f, \Phi(\Phi^*\Phi)^{-1} f\rangle \\
&= \langle \Phi^*\Phi(\Phi^*\Phi)^{-1} f, (\Phi^*\Phi)^{-1} f\rangle \\
&= \langle f, (\Phi^*\Phi)^{-1} f\rangle
\end{aligned}$$

This proves (66).

If $A = B$, then

$$\langle \Phi^*\Phi f, f\rangle = A\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

Thus the spectrum of $\Phi^*\Phi$ is only A, and we have $\Phi^*\Phi = AI$. It follows that $\widetilde{\phi}_\gamma = (\Phi^*\Phi)^{-1}\phi_\gamma = A^{-1}\phi_\gamma$. □

This theorem proves that one way to reconstruct a signal f from its frame coefficients $\Phi f(\gamma) = \langle f, \phi_\gamma\rangle$ is to use the dual frame $\widetilde{\phi}_\gamma$; equivalently, the synthesis coefficients of f in $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ are the dual frame coefficients $\widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma\rangle$. If the frame is tight, then we have the simple reconstruction formula:

$$f = \frac{1}{A}\sum_{\gamma\in\Gamma} \langle f, \phi_\gamma\rangle \phi_\gamma$$

which mirrors the reconstruction of a signal f in an orthonormal basis, except for the factor of A^{-1}.

If $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ is a Riesz basis then the dictionary atoms are linearly independent, which implies that $\text{image}(\Phi) = \boldsymbol{\ell}^2(\Gamma)$; therefore the dual frame $\widetilde{\mathcal{D}} = \{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ is also a Riesz basis. Inserting $f = \phi_n$ into (65) yields:

$$\phi_n = \sum_{\gamma\in\Gamma} \langle \phi_n, \widetilde{\phi}_\gamma\rangle \phi_\gamma$$

The linear independence of $\mathcal{D}$ implies that the only expansion of ϕ_n in $\mathcal{D}$ is the trivial expansion $\phi_n = \phi_n$, which implies that

$$\langle \phi_n, \widetilde{\phi}_\gamma\rangle = \begin{cases} 1 & n = \gamma \\ 0 & n \neq \gamma \end{cases}$$

Thus the frame and dual frame are *biorthogonal bases* for $\mathcal{H}$. Furthermore, if the Riesz basis is normalized so that $\|\phi_\gamma\| = 1$ for all $\gamma \in \Gamma$, then using the dual frame bounds (66) and the biorthogonality we have:

$$\frac{1}{B} = \frac{1}{B}\|\phi_n\|^2 \leq \sum_{\gamma\in\Gamma} |\langle \phi_n, \widetilde{\phi}_\gamma\rangle|^2 = 1 \leq \frac{1}{A}\|\phi_n\|^2 = \frac{1}{A}$$

This shows that

$$A \leq 1 \leq B$$

for a Riesz basis with normalized atoms.

Exercise 59. Read Section 5.1.2 of *A Wavelet Tour of Signal Processing.*

Exercise 60. Prove that if $K \neq 0$, then

$$\mathcal{D} = \left\{\phi_n(t) = e^{2\pi int/K}\right\}_{n\in\mathbb{Z}}$$

is a tight frame for $\mathbf{L}^2[0,1]$. Compute the frame bound.

Exercise 61. Prove that a finite set of N vectors $\{\phi_n\}_{1\leq n\leq N}$ is always a frame for the space $\mathbf{V}$ defined by:

$$\mathbf{V} = \text{span}\{\phi_n\}_{1\leq n\leq N}$$

Exercise 62. Let $\phi_p \in \mathbb{R}^N$ be defined as:

$$\phi_p[n] = \delta[(n-p) \bmod N] - \delta[(n-p-1) \bmod N], \quad 0 \leq p < N$$

and define $\mathbf{V}$ as:

$$\mathbf{V} = \left\{ f \in \mathbb{R}^N : \sum_{n=0}^{N-1} f[n] = 0 \right\}$$

Prove that the dictionary $\mathcal{D} = \{\phi_p\}_{0\leq p<N}$ is a translation invariant frame for $\mathbf{V}$; that is, prove there exists some filter $h \in \mathbb{R}^N$ such that $(\Phi f)(p) = (f \circledast h)(p)$ for all $f \in \mathbf{V}$, where $(\Phi f)(p) = \langle f, \phi_p\rangle$ is the frame analysis operator. Compute the frame bounds. Is it a numerically stable frame when N is large?

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