Math 994-002: Applied and Computational Harmonic Analysis, MSU, Spring 2020

Lecture 25 & 26: Frame Projections and Semi-Discrete Frames

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6.1.3 Dual Frame Analysis and Synthesis Computations

Section 5.1.3 of A Wavelet Tour of Signal Processing.

To compress and denoise a signal f we will project the signal onto a closed subspace $\mathbf{V} \subset \mathcal{H}$ that is generated from the span of a subset dictionary atoms from a larger dictionary. We thus need to study projections onto $\mathbf{V}$. As is well known from linear algebra, the best linear approximation of $f \in \mathcal{H}$ in $\mathbf{V}$ is the orthogonal projection of f onto $\mathbf{V}$. To make clear the setup, we let $\mathcal{D} = \{\phi_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{H}$ be a dictionary in $\mathcal{H}$, but which is a frame *only on* $\mathbf{V}$, i.e.,

$$A\|g\|^2 \leq \sum_{\gamma \in \Gamma} |\langle g, \phi_\gamma \rangle|^2 \leq B\|g\|^2, \quad \forall\, g \in \mathbf{V}$$

The analysis operator Φ is still defined on all of $\mathcal{H}$, but it may not behave "nicely" off of $\mathbf{V}$. The next theorem shows how to compute the orthogonal projection of $f \in \mathcal{H}$ onto $\mathbf{V}$ with the dual frame.

Theorem 6.6. *Let $\mathcal{D} = \{\phi_\gamma\}_{\gamma \in \Gamma}$ be a frame for $\mathbf{V} \subset \mathcal{H}$, and $\widetilde{\mathcal{D}} = \{\widetilde{\phi}_\gamma\}_{\gamma \in \Gamma}$ its dual frame on $\mathbf{V}$. The orthogonal projection of $f \in \mathcal{H}$ onto $\mathbf{V}$ is*

$$P_{\mathbf{V}} f = \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma = \sum_{\gamma \in \Gamma} \langle f, \widetilde{\phi}_\gamma \rangle \phi_\gamma \tag{70}$$

Proof. To show that $P_{\mathbf{V}}$ is a projection, we must show that $P_{\mathbf{V}} g = g$ for all $g \in \mathbf{V}$. But since $\mathcal{D}$ is a frame for $\mathbf{V}$, we have the synthesis formula given by (65) which proves that $P_{\mathbf{V}} g = g$ for all $g \in \mathbf{V}$.

To show that $P_{\mathbf{V}}$ is an orthogonal projection, we must verify that

$$\langle f - P_{\mathbf{V}} f, \phi_n \rangle = 0, \quad \forall\, n \in \Gamma$$

Note that (65) implies that

$$\phi_n = \sum_{\gamma \in \Gamma} \langle \phi_n, \widetilde{\phi}_\gamma \rangle \phi_\gamma$$

Therefore we compute:

$$\begin{aligned}
\langle f - P_{\mathbf{V}} f, \phi_n \rangle &= \langle f - \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma, \phi_n \rangle \\
&= \langle f, \phi_n \rangle - \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \langle \widetilde{\phi}_\gamma, \phi_n \rangle \\
&= \langle f, \phi_n \rangle - \sum_{\gamma\in\Gamma} \langle f, \langle \widetilde{\phi}_\gamma, \phi_n \rangle^* \phi_\gamma \rangle \\
&= \langle f, \phi_n - \sum_{\gamma\in\Gamma} \langle \widetilde{\phi}_\gamma, \phi_n \rangle^* \phi_\gamma \rangle \\
&= \langle f, \phi_n - \sum_{\gamma\in\Gamma} \langle \phi_n, \widetilde{\phi}_\gamma \rangle \phi_\gamma \rangle \\
&= \langle f, \phi_n - \phi_n \rangle = 0
\end{aligned}$$

□

Since $\mathcal{D}$ is a frame for a subspace $\mathbf{V} \subset \mathcal{H}$, Φ is only invertible on this subspace and the definition of the pseudo-inverse is now:

$$\Phi^\dagger \Phi f = f, \ \ \forall\, f \in \mathbf{V} \quad \text{and} \quad \Phi^\dagger a = 0, \ \ \forall\, a \in \text{image}(\Phi)^\perp$$

Let $\Phi_{\mathbf{V}}$ be the restriction of the frame analysis operator to $\mathbf{V}$. The operator $\Phi^* \Phi_{\mathbf{V}}$ is invertible on $\mathbf{V}$ and we write $(\Phi^*\Phi_{\mathbf{V}})^{-1}$ as its inverse on $\mathbf{V}$. One can verify that

$$\Phi^\dagger = (\Phi^*\Phi_{\mathbf{V}})^{-1}\Phi^* = \widetilde{\Phi}^*$$

Let $f \in \mathcal{H}$. Theorem 6.6 and (70) give two ways in which to compute orthogonal projections onto $\mathbf{V}$. In a dual synthesis scenario, the orthogonal projection $P_{\mathbf{V}} f$ is computed from the frame analysis coefficients with the dual frame synthesis operator:

$$P_{\mathbf{V}} f = \widetilde{\Phi}^* \Phi f = \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma \tag{71}$$

If the frame $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ does not depend on the signal f, then the dual frame vectors are precomputed:

$$\widetilde{\phi}_\gamma = (\Phi^*\Phi_{\mathbf{V}})^{-1}\phi_\gamma$$

and the signal $P_{\mathbf{V}} f$ is synthesized with (71).

However, in many applications the frame vectors depend on the signal f. In this case the dual frame vectors $\widetilde{\phi}_\gamma$ cannot be computed in advance, and it is highly inefficient to compute them directly for each new signal f. In this case, we have already computed Φf and we want to compute $P_{\mathbf{V}} f$. We compute first:

$$y = \Phi^*\Phi f = \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \phi_\gamma \in \mathbf{V}$$

Let L be the linear operator defined as

$$Lh = \Phi^* \Phi_{\mathbf{V}} h, \quad \forall\, h \in \mathbf{V}$$

We then compute $P_{\mathbf{V}} f$ via:

$$L^{-1} y = (\Phi^* \Phi_{\mathbf{V}})^{-1} \Phi^* \Phi f = \widetilde{\Phi}^* \Phi f = P_{\mathbf{V}} f$$

We have already encountered several situations which would lead to something similar to the above scenario. For example, when we studied instantaneous frequencies we focused on the ridge points of either the windowed Fourier transform $Sf(u, \xi)$ or the wavelet transform $Wf(u, s)$. While these are not frames according to our current definition (since the index set (u, ξ) or (u, s) is uncountable), this is something we will remedy shortly. The subspace $\mathbf{V}$ then depends on the signal f since it is the subspace of $\mathcal{H}$ generated by the span of the $g_{u,\xi}$ or the $\psi_{u,s}$ that correspond to the ridge points of f in either the windowed Fourier or wavelet representation. Computing $P_{\mathbf{V}} f$ then synthesizes a signal $\widetilde{f}$ from only the ridge information of f. One can do something similar (and we will in a bit) when analyzing signals with isolated singularities and generating $\mathbf{V}$ as the span of the $\psi_{u,s}$ that correspond to the wavelet modulus maxima. As we shall see the synthesized signal $\widetilde{f} = P_{\mathbf{V}} f \approx f$, thus indicating that these local maxima points carry the majority of information in such signals.

The alternate scenario is a dual analysis, in which $P_{\mathbf{V}} f$ is computed as

$$P_{\mathbf{V}} f = \Phi^* \widetilde{\Phi} f = \sum_{\gamma \in \Gamma} \langle f, \widetilde{\phi}_\gamma \rangle \phi_\gamma$$

Similarly to before, if Φ does not depend upon f, then the dual frame vectors $\widetilde{\phi}_\gamma$ can be precomputed.

It is also possible in this case to view $\mathcal{D} = \{\phi_\gamma\}_{\gamma \in \Gamma}$ as a subset of a larger frame, which has been obtained by solving for a sparse approximation of f in the larger frame.

When $\mathcal{D}$ depends on f, we again circumvent computing the dual frame directly. Let

$$a[\gamma] = \widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma \rangle$$

and note that

$$P_{\mathbf{V}} f = \Phi^* a = \sum_{\gamma \in \Gamma} a[\gamma] \phi_\gamma$$

Since $\Phi P_{\mathbf{V}} f = \Phi f$, we have that

$$\Phi \Phi^* a = \Phi f$$

Let $\Phi^*_{\mathrm{Im}(\Phi)}$ be the restriction of Φ^* to image(Φ). Since $\Phi \Phi^*_{\mathrm{Im}(\Phi)}$ is invertible on image(Φ), we have

$$a = (\Phi \Phi^*_{\mathrm{Im}(\Phi)})^{-1} \Phi f$$

Notice that a is obtained by computing $a = L^{-1} y$, where in this case $y = \Phi f$ and $L = \Phi \Phi^*_{\mathrm{Im}(\Phi)}$.

Exercise 63. Read Section 5.1.3 of *A Wavelet Tour of Signal Processing.*

Exercise 64. Read Section 5.1.4 of *A Wavelet Tour of Signal Processing.*

6.1.4 Translation Invariant Frames

Section 5.1.5 of A Wavelet Tour of Signal Processing.

Let $\{\phi_\gamma\}_{\gamma\in\Gamma} \subset \mathbf{L}^2(\mathbb{R}^d)$ be a countable family of time frequency atoms. Recall that a translation invariant dictionary is a dictionary $\mathcal{D}$ of the form

$$\mathcal{D} = \{\phi_{u,\gamma}\}_{u\in\mathbb{R},\,\gamma\in\Gamma}$$

where

$$\phi_{u,\gamma}(x) = \phi_\gamma(x-u)$$

The analysis operator associated to $\mathcal{D}$ acts upon $f \in \mathbf{L}^2(\mathbb{R}^d)$ and is defined as

$$\Phi f(u,\gamma) = \langle f, \phi_{u,\gamma}\rangle = f * \bar{\phi}_\gamma(u), \quad \bar{\phi}_\gamma(x) = \phi_\gamma^*(-x)$$

Since the index set of $\mathcal{D}$ is $\mathbb{R}^d \times \Gamma$ is not countable, it is thus not strictly speaking a frame by the definition we have utilized up to this point. However, we can consider the energy of the transform $\Phi f(u,\gamma)$, which is defined as

$$\|\Phi f\|^2 = \sum_{\gamma\in\Gamma} \|\Phi f(\cdot,\gamma)\|_2^2 = \sum_{\gamma\in\Gamma} \int |\Phi f(u,\gamma)|^2\, du$$

If there exist $0 < A \leq B < \infty$ such that

$$A\|f\|_2^2 \leq \sum_{\gamma\in\Gamma} \|\Phi f(\cdot,\gamma)\|_2^2 = \sum_{\gamma\in\Gamma} \|f * \bar{\phi}_\gamma\|_2^2 \leq B\|f\|_2^2 \tag{72}$$

then all of the frame theory results we have studied thus far still apply. We will refer to such dictionaries as *semi-discrete frames*, since their index set is the cross product of $\mathbb{R}^d$ and Γ, where Γ is discrete but of course $\mathbb{R}^d$ is not. The next theorem shows that the semi-discrete frame condition (72) is equivalent to a condition on the Fourier transforms of the generators ϕ_γ.

Theorem 6.7. *Let $\{\phi_\gamma\}_{\gamma\in\Gamma} \subset \mathbf{L}^2(\mathbb{R}^d)$ be a family of generator functions. Then there exist $0 < A \leq B < \infty$ such that*

$$A \leq \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2 \leq B, \quad \textit{for almost every } \omega \in \mathbb{R}^d, \tag{73}$$

if and only if $\mathcal{D} = \{\phi_{u,\gamma}\}_{u\in\mathbb{R}^d,\,\gamma\in\Gamma}$ is a semi-discrete frame with frame bounds A and B. Any family $\{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ that satisfies

$$\sum_{\gamma\in\Gamma} \widehat{\phi}_\gamma^*(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = 1$$

defines a left inverse

$$f = \sum_{\gamma\in\Gamma} \Phi f(\cdot,\gamma) * \widetilde{\phi}_\gamma = \sum_{\gamma\in\Gamma} f * \bar{\phi}_\gamma * \widetilde{\phi}_\gamma$$

and are thus the generators of the dual frame. They are defined in frequency as

$$\widehat{\tilde{\phi}}_\gamma(\omega) = \frac{\widehat{\phi}_\gamma(\omega)}{\sum_{n\in\Gamma} |\widehat{\phi}_n(\omega)|^2}$$

Proof. Let $H : \mathbf{L}^2(\mathbb{R}^d) \to \mathbf{L}^2(\mathbb{R}^d)$ be defined as $Hf = f * \bar{h}$ for some filter h, where $\bar{h}(x) = h^*(-x)$. We first prove that $H^* g = g * h$. Indeed, using the Parseval formula (Theorem 2.12) and the convolution formula we have:

$$\begin{aligned}
\langle g, Hf \rangle &= \int_{\mathbb{R}^d} g(x)(f * \bar{h})^*(x)\, dx \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{g}(\omega)\widehat{f}^*(\omega)\widehat{\bar{h}}^*(\omega)\, d\omega \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{g}(\omega)\widehat{h}(\omega)\widehat{f}^*(\omega)\, d\omega \\
&= \int_{\mathbb{R}^d} g * h(x) f^*(x)\, dx \\
&= \langle g * h, f \rangle
\end{aligned}$$

Now assume that $\mathcal{D}$ is a semi-discrete frame with frame bounds A and B, and let $\Phi_\gamma f = \Phi f(\cdot, \gamma) = f * \bar{\phi}_\gamma$. Since $\mathcal{D}$ is a semi-discrete frame, each $\Phi_\gamma : \mathbf{L}^2(\mathbb{R}^d) \to \mathbf{L}^2(\mathbb{R}^d)$ and by the above computation $\Phi_\gamma^* g = g * \phi_\gamma$. The analysis operator is $\Phi : \mathbf{L}^2(\mathbb{R}^d) \to \boldsymbol{\ell}^2(\Gamma, \mathbf{L}^2(\mathbb{R}^d))$ which can be written as $\Phi f = (\Phi_\gamma f)_{\gamma\in\Gamma}$. Let $G = (g_\gamma)_{\gamma\in\Gamma} \in \boldsymbol{\ell}^2(\Gamma, \mathbf{L}^2(\mathbb{R}^d))$ and now compute the adjoint of Φ:

$$\begin{aligned}
\langle G, \Phi f \rangle &= \sum_{\gamma\in\Gamma} \langle g_\gamma, \Phi_\gamma f \rangle \\
&= \sum_{\gamma\in\Gamma} \langle \Phi_\gamma^* g_\gamma, f \rangle \\
&= \left\langle \sum_{\gamma\in\Gamma} \Phi_\gamma^* g_\gamma, f \right\rangle \\
&= \left\langle \sum_{\gamma\in\Gamma} g_\gamma * \phi_\gamma, f \right\rangle
\end{aligned}$$

It follows that

$$\Phi^* G = \sum_{\gamma\in\Gamma} g_\gamma * \phi_\gamma$$

and furthermore

$$\Phi^* \Phi f = \sum_{\gamma\in\Gamma} f * \bar{\phi}_\gamma * \phi_\gamma$$

The semi-discrete frame condition (72) is equivalent to

$$A\|f\|^2 \leq \|\Phi f\|^2 = \langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2$$

We can rewrite $\langle \Phi^*\Phi f, f\rangle$:

$$\begin{aligned}
\langle \Phi^*\Phi f, f\rangle &= \int_{\mathbb{R}^d} \sum_{\gamma\in\Gamma} f * \bar{\phi}_\gamma * \phi_\gamma(x) f^*(x)\, dx \\
&= \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^d} f * \bar{\phi}_\gamma * \phi_\gamma(x) f^*(x)\, dx \\
&= \sum_{\gamma} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\omega)\widehat{\phi}_\gamma^*(\omega)\widehat{\phi}_\gamma(\omega)\widehat{f}^*(\omega)\, d\omega \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \left(\sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\right) d\omega
\end{aligned}$$

Suppose by contradiction there exists $E \subset \mathbb{R}^d$ with finite but nonzero Lebesgue measure, i.e., $0 < |E| < \infty$, and for which

$$\sum_{\gamma} |\widehat{\phi}_\gamma(\omega)|^2 > B, \quad \forall\, \omega \in E$$

Let $\widehat{f}(\omega) = (2\pi)^{d/2}\mathbf{1}_E(\omega)$. We have that $\|\widehat{f}\|^2 = (2\pi)^d|E|$ and thus $f \in \mathbf{L}^2(\mathbb{R}^d)$ with $\|f\|^2 = |E|$. But then

$$\begin{aligned}
\langle \Phi^*\Phi f, f\rangle &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (2\pi)^d \mathbf{1}_E(\omega) \left(\sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\right) d\omega \\
&> B \int_E d\omega = B|E| = B\|f\|^2
\end{aligned}$$

which contradicts $\langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2$. A similar argument proves the lower bound, and thus we have shown that

$$\text{for a.e. } \omega \in \mathbb{R}^d, \quad A \leq \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2 \leq B$$

Now assume that (73) holds. Let $f \in \mathbf{L}^2(\mathbb{R}^d)$ and multiply through by $(2\pi)^{-d}|\widehat{f}(\omega)|^2$ and integrate over $\mathbb{R}^d$ to obtain:

$$\frac{A}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2\, d\omega \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\, d\omega \leq \frac{B}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2\, d\omega$$

which is equivalent to

$$A\|f\|^2 \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\, d\omega \leq B\|f\|^2 \tag{74}$$

We rewrite the inner part:

$$\begin{aligned}
\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f}(\omega)|^2\sum_{\gamma\in\Gamma}|\widehat{\phi}_\gamma(\omega)|^2\,d\omega &= \sum_{\gamma\in\Gamma}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\omega)\widehat{\phi}_\gamma^*(\omega)\widehat{f}^*(\omega)\widehat{\phi}_\gamma(\omega)\,d\omega \\
&= \sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} f*\bar{\phi}_\gamma(x)(f*\bar{\phi}_\gamma)^*(x)\,dx \\
&= \sum_{\gamma\in\Gamma}\|f*\bar{\phi}_\gamma\|^2 \\
&= \sum_{\gamma\in\Gamma}\|\Phi f(\cdot,\gamma)\|^2
\end{aligned}$$

Plugging this into (74) proves that $\mathcal{D}$ is a semi-discrete frame.

Now let $\{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ be a family of functions that satisfies

$$\sum_{\gamma\in\Gamma}\widehat{\phi}_\gamma^*(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = 1 \tag{75}$$

First, it is clear that such functions are defined in frequency as:

$$\widehat{\widetilde{\phi}}(\omega) = \frac{\widehat{\phi}_\gamma(\omega)}{\sum_{n\in\Gamma}|\widehat{\phi}_n(\omega)|^2} \tag{76}$$

by simply plugging (76) into the left hand side of (75) and verifying that the sum is equal to one. Now define

$$g(x) = \sum_{\gamma\in\Gamma}\Phi(\cdot,\gamma)*\widetilde{\phi}_\gamma(x) = \sum_{\gamma\in\Gamma}f*\bar{\phi}_\gamma*\widetilde{\phi}_\gamma(x)$$

The Fourier transform of g is:

$$\widehat{g}(\omega) = \sum_{\gamma\in\Gamma}\widehat{f}(\omega)\widehat{\phi}_\gamma^*(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = \widehat{f}(\omega)\sum_{\gamma\in\Gamma}\widehat{\phi}_\gamma^*(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = \widehat{f}(\omega)$$

It follows that $g = f$, which completes the proof. □

Exercise 65. Read Section 5.1.5 of *A Wavelet Tour of Signal Processing.*

6.2 Translation Invariant Dyadic Wavelet Transform

Section 5.2 of A Wavelet Tour of Signal Processing.

Recall that a continuous wavelet transform computes

$$Wf(u,s) = \langle f,\psi_{u,s}\rangle = f*\bar{\psi}_s(u), \quad \forall (u,s)\in\mathbb{R}\times(0,\infty) \tag{77}$$

where

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \quad \text{and} \quad \bar{\psi}_s(t) = \frac{1}{\sqrt{s}}\psi^*\left(-\frac{t}{s}\right)$$

The operator W, as defined in (77), does not define an analysis operator of a semi-discrete frame because the scale parameter s takes values over the entire interval $(0, \infty)$, which is not discrete.

A semi-discrete wavelet frame is generated by sampling the scale parameter s along an exponential sequence $\{a^j\}_{j\in\mathbb{Z}}$ for some $a > 1$. In many applications (but not all!), we take $a = 2$. In this case the generating family is $\{\psi_j\}_{j\in\mathbb{Z}}$ with

$$\psi_j(t) = 2^{-j}\psi(2^{-j}t)$$

and the translation invariant dictionary is given by:

$$\mathcal{D} = \{\psi_{u,j}\}_{u\in\mathbb{R},\, j\in\mathbb{Z}}, \quad \psi_{u,j}(t) = \psi_j(t-u) = 2^{-j}\psi(2^{-j}(t-u))$$

The resulting analysis operator defines the dyadic wavelet transform:

$$Wf(u,j) = \langle f, \psi_{u,j}\rangle = f * \bar{\psi}_j(u), \quad \bar{\psi}_j(t) = 2^{-j}\psi^*(-2^{-j}t)$$

Notice that rather than normalizing the dilated wavelets by $2^{-j/2}$, which would be analogous to the normalization $s^{-1/2}$ in the continuous wavelet transform, we normalize by 2^{-j}. This is to simplify the following presentation. It simply means that the normalization preserves the $\mathbf{L}^1$ norm of ψ as opposed to the $\mathbf{L}^2$ norm, that is, $\|\psi_j\|_1 = \|\psi\|_1$. Notice as well that $\widehat{\psi}_j(\omega) = \widehat{\psi}(2^j\omega)$ with this normalization.

Applying Theorem 6.7 shows that $\mathcal{D}$ is a semi-discrete frame if and only if there exists $0 < A \leq B < 0$ such that

$$A \leq \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \leq B, \quad \forall \omega \in \mathbb{R}\setminus\{0\} \tag{78}$$

In this case $W : \mathbf{L}^2(\mathbb{R}) \to \boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}))$ when the scales are restricted to $s = 2^j$. Notice that if ψ is a complex analytic wavelet (meaning that $\widehat{\psi}(\omega) = 0$ for all $\omega \leq 0$), then it is impossible for (78) to hold. We will come back to this in a bit. For now assume that ψ is a real valued wavelet. The standard semi-discrete frame condition, which is equivalent to (78), is written as:

$$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} \|f * \bar{\psi}_j\|_2^2 \leq B\|f\|_2^2$$

Equation (78) shows that if the frequency axis is completely covered by dilated dyadic wavelets, then a dyadic wavelet transform defines a complete and stable representation of $f \in \mathbf{L}^2(\mathbb{R})$; see Figure 32.

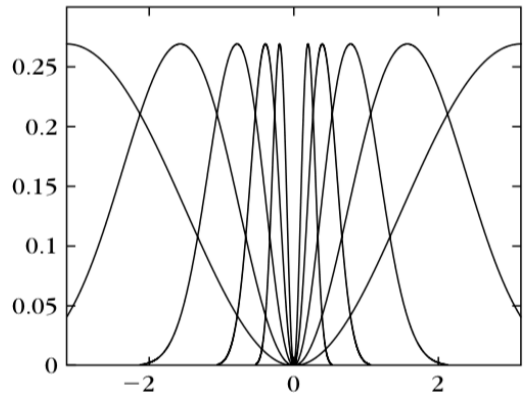

Figure 32: The squared Fourier transform modulus $|\widehat{\psi}(2^j\omega)|^2$ of a real valued spline wavelet, for $1 \leq j \leq 5$ and $\omega \in [-\pi, \pi]$.

Remark 6.8. Recall for the continuous wavelet transform, we had the following admissibility condition for a real valued wavelet:

$$C_\psi = \int_0^{+\infty} \frac{|\widehat{\psi}(\omega)|^2}{\omega} < \infty$$

In fact (78) is closely related to the admissibility condition, as the following calculation shows (let $\omega_0 > 0$):

$$\begin{aligned} C_\psi &= \int_0^{+\infty} \frac{|\widehat{\psi}(\omega)|^2}{\omega}\, d\omega\,, \quad (\text{CoV: } \omega = 2^\lambda \omega_0 \Rightarrow d\omega = (\log 2) 2^\lambda \omega_0 d\lambda) \\ &= \int_{\mathbb{R}} \frac{|\widehat{\psi}(2^\lambda \omega_0)|^2}{2^\lambda \omega_0} (\log 2) 2^\lambda \omega_0 \, d\lambda \\ &= (\log 2) \int_{\mathbb{R}} |\widehat{\psi}(2^\lambda \omega_0)|^2 \, d\lambda \end{aligned}$$

Note that $\omega_0 > 0$ was arbitrary and since $|\widehat{\psi}(\omega)| = |\widehat{\psi}(-\omega)|$ for any ω, in fact it holds for any $\omega_0 \neq 0$. Thus we see (78) is a discrete version of the wavelet admissibility condition. This calculation also explains why switching to a an $\mathbf{L}^1(\mathbb{R})$ normalization for the wavelet is a good idea.

In the case of complex analytic wavelets, one option is to use a larger set of generating wavelets given by:

$$\{\psi_{j,\epsilon}\}_{j\in\mathbb{Z},\, \epsilon\in\{1,-1\}}, \quad \psi_{j,\epsilon}(t) = 2^{-j}\psi(\epsilon 2^{-j} t)$$

In this case for suitably chosen wavelets it is possible for (78) to hold. However, it is unnecessary to double the number of generating wavelets as in the above. Indeed, we can instead replace (78) with

$$2A \leq \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 + \sum_{j\in\mathbb{Z}} |\widehat{\psi}(-2^j\omega)|^2 \leq 2B, \quad \forall\, \omega \in \mathbb{R} \setminus \{0\} \tag{79}$$

which, due to the wavelet ψ being complex analytic, is equivalent to

$$2A \leq \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \leq 2B, \quad \forall\, \omega \in (0,\infty)$$

Let $f \in \mathbf{L}^2(\mathbb{R})$ be real valued and let f_a be the analytic part of f. Recall that $\widehat{f}_a(\omega) = 2\widehat{f}(\omega)$ for $\omega > 0$ and $2\|f\|_2^2 = \|f_a\|_2^2$. Then:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \|f * \bar{\psi}_j\|_2^2 &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |f * \bar{\psi}_j(t)|^2\, dt \\
&= \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 |\widehat{\psi}(2^j\omega)|^2\, d\omega \\
&= \frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}(\omega)|^2 \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2\, d\omega \\
&= \frac{1}{4}\frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}_a(\omega)|^2 \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2\, d\omega \\
&\geq \frac{A}{2}\frac{1}{2\pi} \int_0^{+\infty} |\widehat{f}_a(\omega)|^2\, d\omega \\
&= \frac{A}{2}\|f_a\|_2^2 \\
&= A\|f\|_2^2
\end{aligned}$$

A similar argument shows that $\sum_j \|f * \bar{\psi}_j\|_2^2 \leq B\|f\|_2^2$. Therefore the dyadic wavelet transform with a complex analytic wavelet defines a semi-discrete frame with frame bounds A and B if (79) holds.

Now suppose we only want to compute the dyadic wavelet transform up to a maximum scale 2^j for $j < J$. The lost low frequency information is captured by a single scaling function (or low pass filter) whose Fourier transform is concentrated around the origin. Let $\phi \in \mathbf{L}^2(\mathbb{R})$ be a low pass filter and let $\phi_J(t) = 2^{-J}\phi(2^{-J}t)$ and let ψ be a real valued wavelet. The dyadic wavelet transform in this case is defined as:

$$W_J f = \{f * \bar{\phi}_J(u),\ f * \bar{\psi}_j(u)\}_{u\in\mathbb{R},\, j<J}$$

The operator W_J is the analysis operator of a semi-discrete frame if

$$A \leq |\widehat{\phi}(2^J\omega)|^2 + \sum_{j<J} |\widehat{\psi}(2^j\omega)|^2 \leq B$$

If the family $\{\psi_j\}_{j\in\mathbb{Z}}$ are the generators of a semi-discrete frame, meaning that (78) holds, then one can define ϕ in frequency as:

$$|\widehat{\phi}(\omega)|^2 = \begin{cases} (A+B)/2, & \omega = 0 \\ \sum_{j\geq 0} |\widehat{\psi}(2^j\omega)|^2, & \omega \neq 0 \end{cases}$$

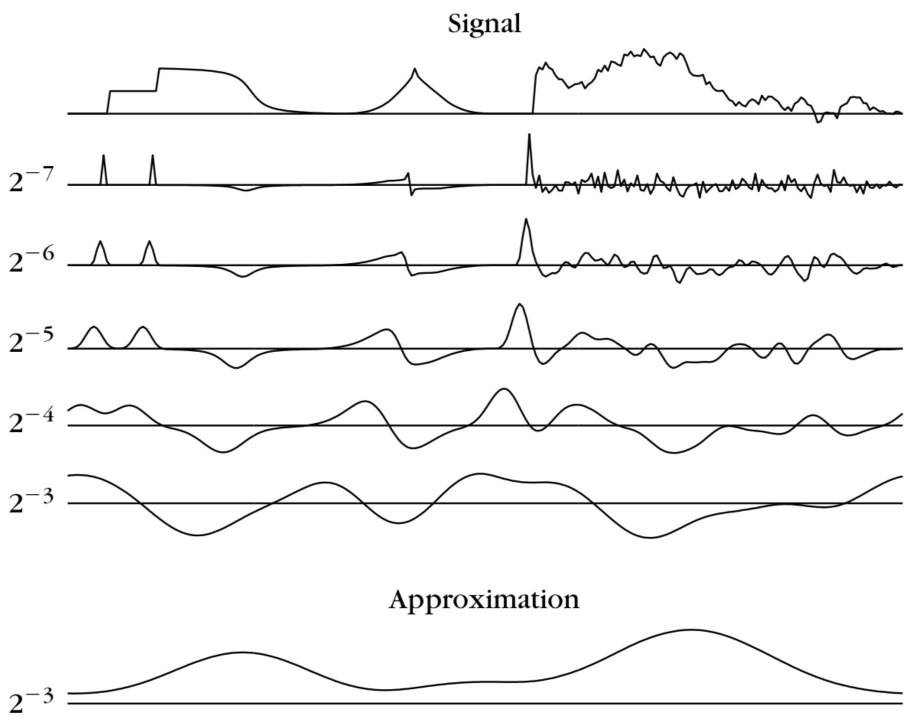

Figure 33: The dyadic wavelet transform $W_J f$ computed with $J = -2$ and $-7 \leq j \leq -3$. The top curve is $f(t)$, the next five curves are $f * \bar{\psi}_j(u)$, and the bottom curve is $f * \bar{\phi}_J$.

Figure 33 plots the dyadic wavelet transform $W_J f$ for the signal f from Figure 13.

A dual wavelet for a semi-discrete dyadic wavelet frame (without scaling function) is computed in frequency as:

$$\widehat{\widetilde{\psi}}(\omega) = \frac{\widehat{\psi}(\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|^2}$$

and the generators of the dual semi-discrete dictionary are given by the dilations of $\widetilde{\psi}$, namely $\{\widetilde{\psi}_j\}_{j\in\mathbb{Z}}$. From this definition it follows that the Fourier transform of $\widetilde{\psi}_j$ satisfies:

$$\widehat{\widetilde{\psi}}_j(\omega) = \widehat{\widetilde{\psi}}(2^j\omega) = \frac{\widehat{\psi}(2^j\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^{j+k}\omega)|^2} = \frac{\widehat{\psi}(2^j\omega)}{\sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|^2}$$

We thus have

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}_j^*(\omega)\widehat{\widetilde{\psi}}_j(\omega) = \sum_{j\in\mathbb{Z}} \widehat{\psi}^*(2^j\omega)\widehat{\widetilde{\psi}}(2^j\omega) = 1, \quad \forall\, \omega \in \mathbb{R}\setminus\{0\}$$

and so by Theorem 6.7 the following reconstruction formula holds:

$$f(t) = \sum_{j\in\mathbb{Z}} f * \bar{\psi}_j * \widetilde{\psi}_j(t)$$

Things are simplified when the semi-discrete dyadic wavelet frame is tight. In this case

$$\widetilde{\psi}_{u,j}(t) = \frac{1}{A}\psi_{u,j}(t) = \frac{1}{A}2^{-j}\psi(2^{-j}(t-u))$$

and signal synthesis is computed as:

$$f(t) = \frac{1}{A}\sum_{j\in\mathbb{Z}} f * \bar{\psi}_j * \psi_j(t)$$

Notice that we must then have

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 = A\,, \quad \forall\, \omega \in \mathbb{R}\setminus\{0\}$$

if the wavelet ψ is real valued (with a similar condition for complex analytic wavelets). This is a Littlewood-Paley type condition, and implies the Fourier transforms of the dilations of the wavelet ψ evenly cover the frequency axis.

Exercise 66. Read Section 5.2 of *A Wavelet Tour of Signal Processing.*

Exercise 67. Read Section 5.3 of *A Wavelet Tour of Signal Processing.*

Exercise 68. Read Section 5.4 of *A Wavelet Tour of Signal Processing.*

Exercise 69. Let h be a filter with $\widehat{h}(0) = \sqrt{2}$ and let $\phi \in \mathbf{L}^2(\mathbb{R})$ be a low pass filter with the following Fourier transform:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2}} \widehat{h}(\omega/2) \widehat{\phi}(\omega/2)$$

Let g be a filter with $\widehat{g}(0) = 0$ and let ψ be a wavelet with Fourier transform:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2}} \widehat{g}(\omega/2) \widehat{\phi}(\omega/2)$$

Prove that if there exist $0 < A \leq B < \infty$ such that

$$A(2 - |\widehat{h}(\omega)|^2) \leq |\widehat{g}(\omega)|^2 \leq B(2 - |\widehat{h}(\omega)|^2)$$

then the family $\{\psi_j\}_{j \in \mathbb{Z}}$ are the generators of a semi-discrete frame.

Exercise 70. Let $X = (X(t))_{t \in \mathbb{R}}$ be a second order stationary stochastic process with continuous sample paths. Let ψ be a real valued, continuous, compactly supported wavelet for which

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi}(2^j \omega)|^2 = 1 \,, \quad \forall\, \omega \neq 0$$

Prove:

$$\sum_{j \in \mathbb{Z}} \mathbb{E}\left[|X * \overline{\psi}_j(t)|^2 \right] = \mathrm{Var}_X(0) = \mathbb{E}[(X(0) - m_X)^2] \,, \quad \forall\, t \in \mathbb{R}$$

Exercise 71. This exercise is about representations of signals f that are invariant to translation of f.

(a) Let $f \in \mathbf{L}^2(\mathbb{R})$ and let $\phi \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{C}^1(\mathbb{R})$ be a low pass filter with $\phi' \in \mathbf{L}^1(\mathbb{R})$. Let $f_u(t) = f(t-u)$ be the translation of f by u. Prove there exists a universal constant $C > 0$ such that:

$$\|f * \phi_J - f_u * \phi_J\|_2 \leq C 2^{-J} |u| \|\phi'\|_1 \|f\|_2$$

(b) Let ϕ be as in part (a) and suppose $\psi \in \mathbf{L}^1(\mathbb{R})$ is a wavelet for which

$$|\widehat{\phi}(2^J \omega)|^2 + \sum_{j < J} |\widehat{\psi}(2^j \omega)|^2 = 1 \,, \quad \forall\, \omega \in \mathbb{R}$$

Part (a) shows that $f * \phi_J$ is a representation of f that is invariant to translations of f so long as $|u| \ll 2^J$. However, $f * \phi_J$ only keeps the low frequencies of f. A representation that keeps more information from f is:

$$S_J f = \{ f * \phi_J \,,\, |f * \psi_j| * \phi_J : j < J \} \in \boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}))$$

Prove this representation is also translation invariant in the same sense, meaning there exists a constant $C > 0$ such that:

$$\|S_J f - S_J f_u\|_{\boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}))} \leq C 2^{-J} |u| \|\phi'\|_1 \|f\|_2$$

Exercise 72. **THIS EXERCISE IS OPTIONAL! JUST IF YOU WANT AN ADDITIONAL CHALLENGE.** The *Zak transform* maps any $f \in \mathbf{L}^2(\mathbb{R})$ to:

$$Zf(u,\xi) = \sum_{l\in\mathbb{Z}} e^{2\pi i l\xi} f(u-l)$$

(a) Prove that $Z : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2[0,1]^2$ is a unitary operator, i.e. show that

$$\int_{\mathbb{R}} f(t)g^*(t)\,dt = \int_0^1 \int_0^1 Zf(u,\xi)Zg^*(u,\xi)\,du\,d\xi$$

One approach is the following: Let $g(t) = \mathbf{1}_{[0,1]}(t)$ and consider

$$\mathcal{B} = \{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}, \quad g_{n,k}(t) = g(t-n)e^{2\pi i k t}$$

Verify that $\mathcal{B}$ is an orthonormal basis for $\mathbf{L}^2(\mathbb{R})$, and then show that $\{Zg_{n,k}\}_{(n,k)\in\mathbb{Z}^2}$ is an orthonormal basis for $\mathbf{L}^2[0,1]^2$.

(b) Prove that the inverse Zak transform is defined by:

$$Z^{-1}h(u) = \int_0^1 h(u,\xi)\,d\xi, \quad \forall\, h \in \mathbf{L}^2[0,1]^2$$

(c) Now let $g \in \mathbf{L}^2(\mathbb{R})$ be arbitrary and consider

$$\mathcal{D} = \{g_{n,k}\}_{(n,k)\in\mathbb{Z}^2}, \quad g_{n,k}(t) = g(t-n)e^{2\pi i k t}$$

Prove that $\mathcal{D}$ is a frame for $\mathbf{L}^2(\mathbb{R})$ with frame bounds $0 < A \leq B < \infty$ if and only if

$$A \leq |Zg(u,\xi)|^2 \leq B, \quad \forall\,(u,\xi) \in [0,1]^2 \tag{80}$$

(d) Prove that if (80) holds, then the dual window $\widetilde{g}$ of the dual frame $\widetilde{D}$ is defined by

$$Z\widetilde{g}(u,\xi) = \frac{1}{Zg^*(u,\xi)}$$

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