Abstract

The scattering transform is a multilayered wavelet-based deep learning architecture that acts as a model of convolutional neural networks. Recently, several works have introduced generalizations of the scattering transform for non-Euclidean settings such as graphs. Our work builds upon these constructions by introducing windowed and non-windowed graph scattering transforms based upon a very general class of asymmetric wavelets. We show that these asymmetric graph scattering transforms have many of the same theoretical guarantees as their symmetric counterparts. This work helps bridge the gap between scattering and other graph neural networks by introducing a large family of networks with provable stability and invariance guarantees. This lays the groundwork for future deep learning architectures for graph-structured data that have learned filters and also provably have desirable theoretical properties.

1 Introduction

The scattering transform is a wavelet-based model of convolutional neural networks (CNNs), introduced for signals defined on $\mathbb{R}^n$ by S. Mallat in [11]. Like the front end of a CNN, the scattering transform produces a representation of an inputted signal through an alternating cascade of filter convolutions and pointwise nonlinearities. It differs from CNNs in two respects: i) it uses predesigned, wavelet filters rather than filters learned through training data, and ii) it uses the complex modulus $|\cdot|$ as its nonlinear activation function rather than more common choices such as the rectified linear unit (ReLU). These differences lead to a network which provably has desirable mathematical properties. In particular, the Euclidean scattering transform is: i) nonexpansive on $L^2(\mathbb{R}^n)$, ii) invariant to translations up to a certain scale parameter, and iii) stable to certain diffeomorphisms. In addition to these theoretical properties, the scattering transform has also been used to achieve very good numerical results in fields such as audio processing [1], medical signal processing [4], computer vision [12], and quantum chemistry [10].

While CNNs have proven tremendously effective for a wide variety of machine learning tasks, they typically assume that inputted data has a Euclidean structure. For instance, an image is naturally modeled as a function on $\mathbb{R}^2$. However, many data sets of interest such as social networks, molecules, or surfaces have an intrinsically non-Euclidean structure and are naturally modeled as graphs or manifolds. This has motivated the rise of geometric deep learning, a field which aims to generalize deep learning methods to non-Euclidean settings. In particular, a number of papers have produced versions of the scattering transform for graph [7, 8, 9, 17] and manifold [13] structured data. These constructions seek to provide a mathematical model of geometric deep learning architectures such as graph neural networks in a manner analogous the way that Euclidean scattering transform models CNNs.
In this paper, we will construct two new families of wavelet transforms on a graph \( G \) from asymmetric matrices \( K \) and provide a theoretical analysis of both of these wavelet transforms as well as the windowed and non-windowed scattering transforms constructed from them. Because the matrices \( K \) are in general not symmetric, our wavelet transforms will not be nonexpansive frame analysis operators on the standard inner product space \( L^2(G) \). Instead, they will be nonexpansive on a certain weighted inner product space \( L^2(G, M) \), where \( M \) is an invertible matrix. In important special cases, our matrix \( K \) will be either the lazy random walk matrix \( P \), its transpose \( P^T \), or its symmetric counterpart given by \( T = D^{-1/2}PD^{-1/2} \). In these cases, \( L^2(G, M) \) is a weighted \( L^2 \) space with weights depending on the geometry of \( G \). We will use these wavelets to construct windowed and non-windowed versions of the scattering transform on \( G \). The windowed scattering transform inputs a signal \( x \in L^2(G, M) \) and outputs a sequence of functions which we refer to as the scattering coefficients. The non-windowed scattering transform replaces the low-pass matrix used in the definition of the windowed scattering transform with an averaging operator \( \mu \) and instead outputs a sequence of scalar-valued coefficients. It can be viewed as the limit of the windowed scattering transform as the scale of the low-pass tends to infinity (evaluated at some fixed coordinate \( 0 \leq i \leq n - 1 \)).

Analogously to the Euclidean scattering transform, we will show that the windowed graph scattering transform is: i) nonexpansive on \( L^2(G,M) \), ii) invariant to permutations of the vertices, up to a factor depending on the scale of the low-pass (for certain choices of \( K \)), and iii) stable to graph perturbations. Similarly, we will show that the non-windowed scattering transform is i) Lipschitz continuous on \( L^2(G,M) \), ii) fully invariant to permutations, and iii) stable to graph perturbations.

1.1 Notation and Preliminaries

Let \( G = (V, E, W) \) be a weighted, connected graph consisting of vertices \( V \), edges \( E \), and weights \( W \), with \( |V| = n \) the number of vertices. If \( x = (x(0), \ldots, x(n - 1))^T \) is a signal in \( L^2(G) \), we will identify \( x \) with the corresponding point in \( \mathbb{R}^n \), so that if \( B \) is an \( n \times n \) matrix, the multiplication \( Bx \) is well defined. Let \( A \) denote the weighted adjacency matrix of \( G \), let \( d = (d(0), \ldots, d(n - 1))^T \) be the corresponding weighted degree vector, and let \( D = \text{diag}(d) \). We will let

\[
N := I - D^{-1/2}AD^{-1/2}
\]

be the normalized graph Laplacian, let \( 0 \leq \omega_0 \leq \omega_1 \leq \ldots \leq \omega_{n-1} \leq 2 \) denote the eigenvalues of \( N \), and let \( v_0, \ldots, v_{n-1} \) be an orthonormal eigenbasis for \( L^2(G) \), \( NV_i = \omega_i v_i \). \( N \) may be factored as

\[
N = V\Omega V^T,
\]

where \( \Omega = \text{diag}(\omega_0, \ldots, \omega_{n-1}) \), and \( V \) is the unitary matrix whose \( i \)-th column is \( v_i \). One may check that \( \omega_0 = 0 \) and that we may choose \( v_0 = \frac{d^{1/2}}{||d^{1/2}||_2}, \) where \( d^{1/2} = (d(0)^{1/2}, \ldots, d(n-1)^{1/2})^T \). We note that since we assume \( G \) is connected, it has a positive spectral gap, i.e.

\[
0 = \omega_0 < \omega_1.
\]

Our wavelet transforms will be constructed from the matrix \( T_g \) defined by

\[
T_g := Vg(\Omega)V^T := VA_g V^T,
\]

where \( g : [0, 2] \to [0, 1] \) is some strictly decreasing spectral function such that \( g(0) = 1 \) and \( g(2) = 0 \), and

\[
A_g := \text{diag}(g(\omega_0), \ldots, g(\omega_{n-1})) := \text{diag}(\lambda_0, \ldots, \lambda_{n-1}).
\]

We note that \( 1 = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0 \), where the fact that \( \lambda_1 < \lambda_0 = 1 \) follows from (\ref{eq:01}). When there is no potential for confusion, we will suppress dependence of \( g \) and write \( T \) and \( \Lambda \) in place of \( T_g \) and \( A_g \). As our main example, we will choose \( g(t) := g_\epsilon(t) := 1 - \frac{t^2}{\epsilon^2} \), in which case

\[
T_{g_\epsilon} = I - \frac{1}{2} \left( I - D^{-1/2}AD^{-1/2} \right) = \frac{1}{2} \left( I + D^{-1/2}AD^{-1/2} \right).
\]
In [8], Gama et al. constructed a graph scattering transform using wavelets which are polynomials in $T_{g_{(1)}}$ and in [9], Gao et al. defined a different, but closely related, graph scattering transform from polynomials of the lazy random walk matrix

$$ \mathbf{P} := \mathbf{D}^{1/2} T_{g}, \mathbf{D}^{-1/2} = \frac{1}{2} (\mathbf{I} + \mathbf{AD}^{-1}) .$$

In order to unify and generalize these frameworks we will let $\mathbf{M}$ be an invertible matrix and let $\mathbf{K}$ be the matrix defined by

$$ \mathbf{K} := \mathbf{M}^{-1} \mathbf{TM}. $$

Note that $\mathbf{K}$ depends on the choice of both $g$ and $\mathbf{M}$, and thus includes a very large family of matrices. As important special cases, we note that we may obtain $\mathbf{K} = \mathbf{T}$ by setting $\mathbf{M} = \mathbf{I}$, and we obtain $\mathbf{P}$ and $\mathbf{P}^T$ by setting $g(t) = g_{(1)}(t)$ and letting $\mathbf{M} = \mathbf{D}^{-1/2}$ and $\mathbf{M} = \mathbf{D}^{1/2}$, respectively. In Section 2 we will construct two wavelet transforms $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ from functions of $\mathbf{K}$ and show that these wavelet transforms are non-expansive frame analysis operators on the appropriate Hilbert space. When $\mathbf{M} = \mathbf{I}$ (and therefore $\mathbf{K} = \mathbf{T}$), this Hilbert space will simply be the standard inner product space $L^2(G)$. However, for general $\mathbf{M}$, the matrix $\mathbf{K}$ will not be self-adjoint on $L^2(G)$. This motivates us to introduce the Hilbert space $L^2(G, \mathbf{M})$, of signals defined on $G$ with inner product defined by

$$ \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{M}} = \langle \mathbf{Mx}, \mathbf{My} \rangle_2,$$

where $\langle \cdot, \cdot \rangle_2$ denotes the standard $L^2(G)$ inner product. We note that the norms $\| \mathbf{x} \|^2_{\mathbf{M}} := \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbf{M}}$, and $\| \mathbf{x} \|^2_2 = \langle \mathbf{x}, \mathbf{x} \rangle_2$ are equivalent and that

$$ \frac{1}{\| \mathbf{M}^{-1} \|^2_2} \| \mathbf{x} \|^2_2 \leq \| \mathbf{x} \|^2_{\mathbf{M}} \leq \| \mathbf{M} \|^2_2 \| \mathbf{x} \|^2_2,$$

where for any $n \times n$ matrix $\mathbf{B}$, we shall let $\| \mathbf{B} \|^2_2$ and $\| \mathbf{B} \|^2_{\mathbf{M}}$ denote its operator norms on $L^2(G)$ and $L^2(G, \mathbf{M})$ respectively. The following lemma, which shows that $\mathbf{K}$ is self-adjoint on $L^2(G, \mathbf{M})$, will be useful in studying the frame bounds of the wavelet transforms constructed from $\mathbf{K}$.

**Lemma 1.** $\mathbf{K}$ is self-adjoint on $L^2(G, \mathbf{M})$.

**Proof.** By construction, $\mathbf{T}$ is self-adjoint with respect to the standard inner product. Therefore, for all $\mathbf{x}$ and $\mathbf{y}$ we have

$$ \langle \mathbf{Kx}, \mathbf{y} \rangle_{\mathbf{M}} = \langle \mathbf{M}(\mathbf{M}^{-1} \mathbf{TM})\mathbf{x}, \mathbf{My} \rangle_2 $$

$$ = \langle \mathbf{TMx}, \mathbf{My} \rangle_2 $$

$$ = \langle \mathbf{Tx}, \mathbf{My} \rangle_2 $$

$$ = \langle \mathbf{Mx}, \mathbf{TMy} \rangle_2 $$

$$ = \langle \mathbf{Mx}, \mathbf{M}(\mathbf{M}^{-1} \mathbf{TM})\mathbf{y} \rangle_2 $$

$$ = \langle \mathbf{Mx}, \mathbf{MKy} \rangle_2 $$

$$ = \langle \mathbf{x}, \mathbf{KY} \rangle_{\mathbf{M}}. $$

$\square$

It will frequently be useful to consider the eigenvector decompositions of $\mathbf{T}$ and $\mathbf{K}$. By definition, we have

$$ \mathbf{T} = \mathbf{VAV}^T $$

where $\mathbf{A} = g(\Omega)$ and $\{ \mathbf{v}_0, \ldots, \mathbf{v}_{n-1} \}$ is an orthonormal eigenbasis for $L^2(G)$ with $\mathbf{Tv}_i = \lambda_i \mathbf{v}_i$. Since the matrices $\mathbf{T}$ and $\mathbf{K}$ are similar with $\mathbf{K} = \mathbf{M}^{-1} \mathbf{TM}$, one may use the definition of $\langle \cdot, \cdot \rangle_{\mathbf{M}}$ to verify that the vectors $\{ \mathbf{u}_0, \ldots, \mathbf{u}_{n-1} \}$ defined by

$$ \mathbf{u}_i := \mathbf{M}^{-1}\mathbf{v}_i $$

form an orthonormal eigenbasis for $L^2(G, \mathbf{M})$, with $\mathbf{Ku}_i = \lambda_i \mathbf{u}_i$. One may also verify that

$$ \mathbf{w}_i := \mathbf{Mv}_i $$
is a left-eigenvector of $K$ and $w_i^T K = \lambda_i w_i^T$ for all $0 \leq i \leq n - 1$.

In the following section, we will construct wavelets from polynomials of $p(K)$. For a polynomial, $p(t) = a_k t^k + \ldots + a_1 t + a_0$ and a matrix $B$, we define $p(B)$ by

$$p(B) = a_k B^k + \ldots + a_1 B + a_0 I$$

The following lemma uses (2) to derive a formula for computing polynomials of $K$ and $T$ and relates the operator norms of polynomials of $K$ to polynomials of $T$. It will be useful for studying the wavelet transforms introduced in the following section.

**Lemma 2.** For any polynomial $p$, we have

$$p(T) = V p(\Lambda) V^T \quad \text{and} \quad p(K) = M^{-1} p(T) M = M^{-1} V p(\Lambda) V^T M.$$  \hfill (3)

Consequently, for all $x \in L^2(G, M)$

$$\|p(K) x\|_M = \|p(T) M x\|_2.$$  \hfill (4)

**Proof.** Since $V$ is unitary, $V^{-1} = V^T$, and so it follows from (2) that

$$T^r = V \Lambda^r V^T$$

for all $r \geq 0$. Moreover, since $K = M^{-1} T M$

$$K^r = (M^{-1} T M)^r = M^{-1} T^r M = M^{-1} V \Lambda^r V^T M.$$  \hfill (3)

Linearity now implies (4). (4) follows by recalling that $\|x\|_M = \|M x\|_2$, and noting therefore that for all $x$,

$$\|p(K) x\|_M = \|M(p(T) M x)\|_2 = \|p(T) M x\|_2.$$  \hfill (4)

In light of Lemma 2 for any polynomial $p$, we may define $p(T)^{1/2}$ and $p(K)^{1/2}$ by

$$p(T)^{1/2} := V^T p(\Lambda)^{1/2} V^T \quad \text{and} \quad p(K)^{1/2} = M^{-1} V^T p(\Lambda)^{1/2} V^T M,$$  \hfill (5)

where the square root of the diagonal matrix $p(\Lambda)$ is defined entrywise. We may readily verify that

$$p(T)^{1/2} p(T)^{1/2} = p(T) \quad \text{and} \quad p(K)^{1/2} p(K)^{1/2} = p(K).$$

### 1.2 Related Work

Graph scattering transforms have previously been introduced by Gama, Ribeiro, and Bruna in [7] and [8], by Gao, Wolf, and Hirn in [9], and by Zou and Lerman in [17]. In [17], the authors construct a family of wavelet convolutions using the spectral decomposition of the unnormalized graph Laplacian and define a windowed scattering transform as an iterative series of wavelet convolutions and nonlinearities. They then prove results analogous to Theorems 4 and 6 of this paper for their windowed scattering transform. They also introduce a notion of stability to graph perturbations. However, their notion of graph perturbations is significantly different than the one we consider in Section 4.

In [8], the authors construct a family of wavelets from polynomials of $T$, in the case where $g(t) = g_*(t) = 1 - \frac{t}{2}$, and showed that the resulting non-windowed scattering transform was stable to graph perturbations. These results were then generalized in [7], where the authors introduced a more general class of graph convolutions, constructed from a class of symmetric matrices known as “graph shift operators.” The wavelet transform considered in [8] is nearly identical to the $W^{(2)}$ introduced in Section 2 in the special case where $g(t) = g_*(t)$ and $M = I$, with the only difference being that our wavelet transform includes a low-pass filter.

In [9], wavelets were constructed from the lazy random walk matrix $P = D^{1/2} T D^{-1/2}$. These wavelets are essentially the same as the $W^{(2)}$ in the case where $g(t) = g_*(t)$ and $M = D^{-1/2}$, although similarly to [8], the wavelets in [9] do not use a low-pass filter. In all of these previous works, the authors carry out substantial
numerical experiments and demonstrate that scattering transforms are effective for a variety of graph deep learning tasks.

Our work here is meant to unify and generalize the theory of these previous constructions. Our introduction of the matrix $M$ allows us to obtain wavelets very similar to either [8] or [9] as special cases. Moreover, the introduction of the tight wavelet frame $W(1)$ allows us to produce a network with provable conservation of energy and nonexpansive properties analogous to [17]. To highlight the generality of our setup, we introduce both windowed and non-windowed versions of the scattering transform using general (wavelet) frames and provide a detailed theoretical analysis of both. In the case where $M = I$ (and therefore $K = T$) much of this analysis is quite similar to [8]. However, for general $M$, this matrix $K$ is asymmetric which introduces substantial challenges. While [9] demonstrated that asymmetric wavelets are numerically effective in the case $K = P$, this work is the first to produce a theoretical analysis of graph scattering transforms constructed with asymmetric wavelets.

We believe that the generality of our setup introduces a couple of exciting new avenues for future research. In particular, we have introduced a large class of scattering transforms with provable stability and invariance guarantees. In the future, one might attempt to learn the matrix $M$ or the spectral function $g$ based off of data for improved numerical performance on specific tasks. This could be an important step towards bridging the gap between scattering transforms, which act as a model of neural networks, and other deep learning architectures. We also note that a key difference between our work and [18] is that we use the normalized graph Laplacian whereas they use the unnormalized Laplacian. It is quite likely that asymmetric wavelet transforms similar to ours can be constructed from the spectral decomposition of the unnormalized Laplacian. However, we leave that to future work.

## 2 The Graph Wavelet Transform

In this section, we will construct two graph wavelet transforms based off of the matrix $K = M^{-1}TM$ introduced in Section [11]. In the following sections, we will provide a theoretical analysis of the scattering transforms constructed from each of these wavelet transforms and of their stability properties.

Let $J \geq 0$, and for $0 \leq j \leq J + 1$, let $p_j$ be the polynomial defined by

$$p_j(t) = \begin{cases} 1 - t & \text{if } j = 0, \\ t^{2j-1} - t^{2j} & \text{if } 1 \leq j \leq J, \\ t^{2J} & \text{if } j = J + 1 \end{cases}$$

and let $q_j(t) = p_j(t)^{1/2}$. We note that by construction

$$\sum_{j=0}^{J+1} p_j(t) = \sum_{j=0}^{J+1} q_j(t)^2 = 1 \text{ for all } 0 \leq t \leq 1. \quad (6)$$

Using these functions we define two wavelet transforms by

$$W^{(1)}_j = \left\{ \Psi_j^{(1)}, \Phi_j^{(1)} \right\}_{0 \leq j \leq J} \quad \text{and} \quad W^{(2)}_j = \left\{ \Psi_j^{(2)}, \Phi_j^{(2)} \right\}_{0 \leq j \leq J},$$

where

$$\Psi_j^{(1)} = q_j(K), \quad \Phi_j^{(1)} = q_{J+1}(K), \quad \Psi_j^{(2)} = p_j(K), \quad \text{and} \quad \Phi_j^{(2)} = p_{J+1}(K),$$

and the $q_j(K)$ are defined as in [5]. The next two propositions show $W^{(1)}_j$ is an isometry and $W^{(2)}_j$ is a nonexpansive frame analysis operator on $L^2(G, M)$.

**Proposition 1.** $W^{(1)}_j$ is an isometry from $L^2(G, M)$ to $l^2(L^2(G, M))$. That is, for all $x \in L^2(G, M)$,

$$\left\| W^{(1)}_j x \right\|_{l^2(L^2(G, M))}^2 := \sum_{j=0}^{J} \left\| \Psi_j^{(1)} x \right\|_{M}^2 + \left\| \Phi_j^{(1)} x \right\|_{M}^2 = \|x\|_{M}^2.$$
Proof. Proposition 1 shows $K$ is self-adjoint on $L^2(G, M)$. By Lemma 2 and by (5) we have

$$\Psi_j^{(1)} = q_j(K) = M^{-1} V_q j(\Lambda) V^T M$$

for $0 \leq j \leq J$, and

$$\Phi_j^{(1)} = q_{J+1}(K) = M^{-1} V_q J+1(\Lambda) V^T M.$$  

Thus, $\Psi_0^{(1)}, \ldots, \Psi_J^{(1)}$, and $\Phi_j^{(1)}$ are all self-adjoint on $L^2(G, M)$ and are diagonalized in the same basis. Therefore, lower and upper the frame bounds of $W^{(1)}$ are given by computing

$$\min_{0 \leq i \leq n-1} Q(\lambda_i) \quad \text{and} \quad \max_{0 \leq i \leq n-1} Q(\lambda_i),$$

where $Q(t) := \sum_{j=0}^{J+1} q_j(t)^2$. The proof follows from recalling that by (6), we have that $Q(t) = 1$ uniformly on $0 \leq t \leq 1$, and therefore, $W^{(1)}$ is an isometry.

Proof. By the same reasoning as in the proof of Proposition 1, the frame bounds of $W^{(2)}$ are given by computing

$$\min_{0 \leq i \leq n-1} P(\lambda_i) \quad \text{and} \quad \max_{0 \leq i \leq n-1} P(\lambda_i),$$

where $P(t) = \sum_{j=0}^{J+1} p_j(t)^2$. Since $0 \leq \lambda_i \leq 1$ for all $i$, we have

$$\max_i P(\lambda_i) \leq \max_{[0,1]} \left( \sum_{j=0}^{J+1} p_j(t)^2 \right) \leq \max_{[0,1]} \left( \sum_{j=0}^{J+1} p_j(t) \right)^2 = 1$$

with the middle inequality following from the fact that $p_j(t) \geq 0$ for all $t \in [0,1]$, and the last equality following from (6). For the lower bound, we note that

$$\min_{0 \leq i \leq n-1} P(\lambda_i) \geq \min_{0 \leq i \leq 1} \sum_{j=0}^{J+1} p_j(t)^2 \geq \min_{0 \leq i \leq 1} [p_0(t)^2 + p_{J+1}(t)^2] = \min_{0 \leq i \leq 1} \left[ (1-t)^2 + t^{2J+1} \right] = C_J > 0.$$  

3 The Scattering Transform

In this section, we will construct the scattering transform as a multilayered architecture built off of a frame $W$ such as the wavelet transforms $W^{(1)}$ and $W^{(2)}$ introduced in Section 2. We shall see the scattering transform constructed is a continuous operator on $L^2(G, M)$ whenever $W$ is nonexpansive. We shall also see that it has desirable conservation of energy bounds when $W = W^{(1)}$ due to the fact that $W^{(1)}$ is an isometry. On the other hand, we shall see in the following section that the scattering transform has much stronger stability guarantees when $W = W^{(2)}$. 


3.1 Definitions

Let $G = (V, E, W)$ be a connected weighted graph with $|V| = n$, let $M$ be an invertible matrix, and let $J$ be some indexing set. Assume that

$\mathcal{W} = \{\Psi_j, \Phi\}_{j \in J}$

is a frame on $L^2(G, M)$ such that

$$A\|x\|_M^2 \leq \|\mathcal{W}x\|_{L^2(G, M)}^2 \leq \sum_{j \in J} \|\Psi_j x\|_M^2 + \|\Phi x\|_M^2 \leq B \|x\|_M^2,$$

(7)

for some $0 < A < B < \infty$. In this paper, we are primarily interested in the case where $J = \{0, \ldots, J\}$ and $\mathcal{W}$ is either $\mathcal{W}_j^{(1)}$ or $\mathcal{W}_j^{(2)}$. Therefore, we will think of the matrices $\Psi_j$ as wavelets, and $\Phi$ as a low-pass filter. However, we will define the scattering transform for generic frames in order to highlight the relationship between properties of the scattering transform and of the underlying frame.

Letting $M : L^2(G, M) \to L^2(G, M)$ be the pointwise modulus function $Mx = (|x(0)|, \ldots, |x(n-1)|)$, we define $U : L^2(G, M) \to \ell^2(L^2(G, M))$ by

$$Ux := \{U[j]x : m \geq 0, j = (j_1, \ldots, j_m) \in J^m\}.$$

Here, $J^m$ is the $m$-fold Cartesian product of $J$ with itself, the $U[j]x$ are defined by

$$U[j]x = M\Psi_{j_1} \ldots M\Psi_{j_m}x,$$

for $m \geq 1$, and we declare that $U[j_w]x = x$ when $m = 0$ and $j_w$ is the “empty index.” We then define the windowed and non-windowed scattering transforms, $S : L^2(G, M) \to \ell^2(L^2(G, M))$ and $\overline{S} : L^2(G, M) \to \ell^2$ by

$$Sx = \{S[j]x : m \geq 0, j = (j_1, \ldots, j_m) \in J^m\} \quad \text{and} \quad \overline{S}x = \{\overline{S}[j]x : m \geq 0, j = (j_1, \ldots, j_m) \in J^m\}.$$

where the scattering coefficients $S[j]$ and $\overline{S}[j]$ are defined by

$$S[j]x = \Phi U[j]x \quad \text{and} \quad \overline{S}[j]x = (\mu, U[j]x)_M$$

for some weighting vector $\mu \in L^2(G, M)$. One natural choice is $\mu = (M^T M)^{-1} 1$, where $1$ is the vector of all ones. In this case, one may verify that $\overline{S}[j]x = \|U[j]x\|_1$, and we recover a setup similar to [9]. Another natural choice is $\mu = u_0$, in which case we recover a setup similar to $S$ if we set $M = I$.

In practice, one only uses finitely many scattering coefficients. This motivates us to consider the partial scattering transforms defined for $0 \leq \ell \leq L \leq \infty$ by

$$S^{(L)}_\ell x = \{S[j]x : j = (j_1, \ldots, j_m) \in J^m, \ell \leq m \leq L\}$$

and

$$\overline{S}^{(L)}_\ell x = \{\overline{S}[j]x : j = (j_1, \ldots, j_m) \in J^m, \ell \leq m \leq L\}.$$

3.2 Continuity and Conservation of Energy Properties

The following theorem shows that the windowed scattering transform $S$ is nonexpansive and the non-windowed scattering transform $\overline{S}$ is Lipschitz continuous when $\mathcal{W}$ is either $\mathcal{W}_j^{(1)}$ or $\mathcal{W}_j^{(2)}$ or, more generally, whenever $\mathcal{W}$ is nonexpansive.

**Theorem 1.** If $B \leq 1$ in (7), then the windowed scattering transform $S$ is a nonexpansive operator from $L^2(G, M)$ to $\ell^2(L^2(G, M))$, and the non-windowed scattering transform $\overline{S}$ is a Lipschitz continuous operator from $L^2(G, M)$ to $\ell^2$. Specifically, for all $x, y \in L^2(G, M)$,

$$\|Sx - Sy\|_{\ell^2(L^2(G, M))} \leq \|x - y\|_M,$$

(8)

and

$$\|\overline{S}x - \overline{S}y\|_{\ell^2} \leq \|\mu\|_M \|\Phi^{-1}\|_M \|x - y\|_M.$$

(9)

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The proof of \cite{13} is very similar to analogous results in e.g., \cite{11} and \cite{17}. The proof of \cite{19} uses the relationship $UX = \Phi^{-1}SX$ to show

$$\|SX - SY\|_2^2 \leq \|\mu\|_M \|\Phi^{-1}\|_M \|SX - SY\|_{E^2(L^2(G, M))}.$$  

Full details are provided in Appendix \cite{B}.

The next theorem shows that if $W$ is either of the wavelet transforms constructed in Section \cite{2} then $U$ experiences rapid energy decay. Our arguments use ideas similar to the proof of Proposition 3.3 of \cite{17}, with minor modifications to account for the fact that our wavelet constructions are different. Please see Appendix \cite{C} for a complete proof.

**Theorem 2.** Let $J \geq 0$, let $J := \{0, \ldots, J\}$, and let $W = \{\Psi_j, \Phi\}_{j \in J}$ be either of the wavelet transforms, $W_{J(1)}$ or $W_{J(2)}$, constructed in Section \cite{3}. Then for all $x \in L^2(G, M)$ and all $m \geq 1$

$$\sum_{j \in J^{m+1}} \|U[j]x\|_M^2 \leq \left(1 - \frac{d_{\text{min}}}{\|d\|_1}\right) \sum_{j \in J^m} \|U[j]x\|_M^2. \quad (10)$$

Therefore, for all $m \geq 0$,

$$\sum_{j \in J^{m+1}} \|U[j]x\|_M^2 \leq \left(1 - \frac{d_{\text{min}}}{\|d\|_1}\right)^m \|x\|_M^2. \quad (11)$$

The next theorem shows that if $W = W_{J(1)}$, then the windowed graph scattering transform conserves energy on $L^2(G, M)$. Its proof, which relies on Proposition \cite{1} Theorem \cite{2} and Lemma \cite{5}, is nearly identical to the proof of Theorem 3.1 in \cite{17}. We give a proof in Appendix \cite{D} for the sake of completeness.

**Theorem 3.** Let $J \geq 0$, let $J := \{0, \ldots, J\}$, and let $W = W_{J(1)}$. Then the non-windowed scattering transform is energy preserving, i.e., for all $x \in L^2(G, M)$,

$$\|SX\|_{E^2(L^2(G, M))} = \|x\|_M.$$  

### 3.3 Permutation Invariance and Equivariance

In this section, we will show that both $U$ and the windowed graph scattering transform are permutation equivariant. As a consequence, we will be able to show that the non-windowed scattering transform is permutation invariant and that under certain assumptions the windowed-scattering transform is permutation invariant up to a factor depending on the scale of the low-pass filter.

Let $S_n$ denote the permutation group on $n$ elements, and, for $\Pi \in S_n$, let $G' = \Pi(G)$ be the graph obtained by permuting the vertices of $G$. We define $M'$, which we view as the analog of $M$ associated to $G'$, by

$$M' = \Pi M \Pi^T.$$  

To motivate this definition, we note that if $M$ is the identity, then $M'$ is also the identity, and if $M = D^{1/2}$, the square-root degree matrix, then the square-root degree matrix on $G'$ is given by

$$\Pi D^{1/2} \Pi^T,$$

with a similar formula holding when $M = D^{-1/2}$. We define $W'$ and $\mu'$, to be the frame and the weighting vector on $G'$, corresponding to $W$ and $\mu$, by

$$W' := \Pi \Psi \Pi^T := \{\Pi \Psi_j \Pi^T, \Pi \Phi \Pi^T\}_{j \in J} \quad \text{and} \quad \mu' = \Pi \mu, \quad (12)$$

and we let $S'$ and $\overline{S}$ denote the corresponding windowed and non-windowed scattering transforms on $G'$.

To understand $W'$, we note that the natural analog of $T$ on $G'$ is given by

$$T' = \Pi T \Pi^T.$$
Therefore, Lemma 2 implies that for any polynomial $p$,
\[
p((M')^{-1}TM') = (M')^{-1}p(T')M'
\]
with a similar formula holding $q := p^{1/2}$. Therefore, if $W$ is either of the wavelet transforms $W^{(1)}_j$ or $W^{(2)}_j$, then $W'$ is analogous wavelet transform constructed from $K' := (M')^{-1}T'M'$.

**Theorem 4.** Both $U$ and the windowed scattering transform $S$ are equivariant to permutations. That is, if $\Pi \in S_n$ is any permutation and $W'$ is defined as in (12), then for all $x \in L^2(G, M)$
\[
U'\Pi x = \Pi Ux \quad \text{and} \quad S'\Pi x = \Pi Sx.
\]

**Proof.** Let $\Pi$ be a permutation. Since $\Pi(Mx) = M(\Pi x)$ and $\Pi^T = \Pi^{-1}$, it follows that for all $j \in J$
\[
U'[j]\Pi x = M\Psi'_j\Pi x = M\Pi\Psi_j^T\Pi x = \Pi M\Psi_j x = \Pi U[j]x.
\]
For $j = (j_1, \ldots, j_m)$, we have $U[j] = U[j_1] \cdots U[j_m]$. Therefore, it follows inductively that $U$ is equivariant to permutations. Since $S = \Phi U$, we have that
\[
S'\Pi x = \Phi'U'\Pi x = \Pi \Phi'\Pi^T\Pi Ux = \Pi Sx.
\]
Thus, the windowed scattering transform is permutation equivariant as well. $\square$

**Theorem 5.** The non-windowed scattering transform $\overline{S}$ is fully permutation invariant, i.e., for all permutations $\Pi$ and all $x \in L^2(G, M)$
\[
\overline{S} \Pi x = \overline{S} x.
\]

**Proof.** Since $U$ is permutation equivariant by Theorem 4 and $\mu' = \Pi \mu$, we may use the fact that $M' = \Pi M\Pi^T$ and that $\Pi^T = \Pi^{-1}$ to see that for any $x$ and any $j$,
\[
\overline{S}[j]\Pi x = \langle \mu', U'[j]\Pi x \rangle_{M'} = \langle M'\Pi \mu, M'\Pi U[j]x \rangle_2 = \langle \Pi M \mu, \Pi MU[j]x \rangle_2 = \langle M \mu, MU[j]x \rangle_2 = \overline{S}[j]x.
\]

Next, we will use Theorem 4 to show that if $W$ is either $W^{(1)}_j$ or $W^{(2)}_j$ and $M = D^{1/2}$, then the windowed scattering transform is invariant on $L^2(G, M)$ up to a factor depending on the scale of the low-pass filter. We note that $0 < \lambda_1 < 1$. Therefore, $\lambda_j$ decays exponentially fast as $t \to \infty$, and so if $J$ is large, the right hand side of (13) will be nearly zero. We also recall that if our spectral function is given by $g(t) = g_*(t)$ then this choice of $M$ will imply that $K = P^T$.

**Theorem 6.** Let $M = D^{1/2}$, and let $W$ be either $W^{(1)}_j$ or $W^{(2)}_j$. Then the windowed-scattering transform is permutation invariant up to a factor depending on $J$. Specifically, for all $\Pi \in S_n$ and for all $x \in L^2(G, M)$,
\[
\|S'\Pi x - Sx\|_{(L^2(G, M))} \leq \lambda^J_1 \Pi - I\|_M \left(1 + \frac{\|d\|_{\infty}}{d_{\min}}\right)^{1/2} \|x\|_M, \tag{13}
\]
where $t = 2^{j-1}$ if $W = W^{(1)}_j$ and $t = 2^j$ if $W = W^{(2)}_j$.

**Proof.** By Theorem 4 and the fact that $S = \Phi U$ we see that
\[
\|S'\Pi x - Sx\|_{(L^2(G, M))} = \|\Pi Sx - Sx\|_{(L^2(G, M))} = \|\Pi \Phi U x - \Phi U x\|_{(L^2(G, M))} \leq \|\Pi \Phi - \Phi\|_M \|Ux\|_{(L^2(G, M))}. \tag{14}
\]
Let \( t = 2^{j-1} \) if \( W = W^{(1)} \), and let \( t = 2^j \) if \( W = W^{(2)} \) so that in either case \( \Phi = T^t \). Let \( x \in L^2(G, M) \), \( \| \cdot \|_M \) implies that for any \( y \in L^2(G, M) \)

\[
T^t y = \sum_{i=0}^{n-1} \lambda_i^t (v_i, z) v_i.
\]

Therefore, by Lemma 2 and the relationship \( u_i = M^{-1} v_i \), we have

\[
K^t x = M^{-1} T^t (Mx) = \sum_{i=0}^{n-1} \lambda_i^t (v_i, Mx) M^{-1} v_i = \sum_{i=0}^{n-1} \lambda_i^t (v_i, Mx) u_i.
\]

Since \( v_0 = \frac{d^{1/2}}{\|d\|^{1/2}} \), and \( u_i = M^{-1} v_i \), the assumption that \( M = D^{1/2} \) implies that \( u_0 = \frac{1}{\|d\|^{1/2}} 1 \). Therefore, \( \Pi u_0 = u_0 \), and so

\[
\Pi K^t x - K^t x = \sum_{i=0}^{n-1} \lambda_i^t (v_i, Mx) (\Pi u_i - u_i) = \sum_{i=0}^{n-1} \lambda_i^t (v_i, Mx) (\Pi u_i - u_i) = (\Pi - I) \left( \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx) u_i \right).
\]

Therefore, since \( \{ u_0, \ldots, u_{n-1} \} \) forms an orthonormal basis for \( L^2(G, M) \), we have that by Parseval’s identity

\[
\| \Pi K^t x - K^t x \|_M^2 \leq \| \Pi - I \|_M^2 \left( \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2 u_i \right)
\]

\[
= \| \Pi - I \|_M^2 \left( \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2 \right)
\]

\[
\leq \| \Pi - I \|_M^2 \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2
\]

\[
\leq \| \Pi - I \|_M^2 \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2
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= \| \Pi - I \|_M^2 \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2
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= \| \Pi - I \|_M^2 \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2
\]

\[
= \| \Pi - I \|_M^2 \sum_{i=1}^{n-1} \lambda_i^t (v_i, Mx)^2
\]

(15)

To bound \( \| Ux \|_{L^2(G,M)} \), we note that by Theorem 2,

\[
\| Ux \|_{L^2(G,M)} = \| x \|_M + \left( \sum_{m=1}^{\infty} \sum_{j \in J^m} \| U[j] x \|_M^2 \right)
\]

\[
\leq \| x \|_M + \left( \sum_{m=1}^{\infty} \left( 1 - \frac{1}{n} \frac{d_{\min}}{d_{\infty}} \right) ^m \sum_{j \in J^m} \| U[j] x \|_M^2 \right)
\]

\[
\leq \| x \|_M + \frac{\| d \|_{\infty}}{d_{\min}} \sum_{j \in J^1} \| U[j] x \|_M^2
\]

\[
\leq \left( 1 + \frac{\| d \|_{\infty}}{d_{\min}} \right) \| x \|_M.
\]

where the last inequality uses the fact that the modulus operator is nonexpansive and that \( B \leq 1 \) in (7). Combining this with (14) and (15) completes the proof.

\[
\Box
\]

4 Stability to Graph Perturbations

Let \( G = (V, E, W) \) and \( G' = (V', E', W') \) be weighted graphs with \( |V| = |V'| = n \), and let \( M \) and \( M' \) be invertible matrices. Throughout this section, for any object \( X \) associated to \( G \), we will let \( X' \) denote the analogous object on \( G' \), so e.g., \( D' \) is the degree matrix of \( G' \). Recall that two important examples of
our asymmetric matrix $K$ are when $g(t) = g_*(t) = 1 - \frac{t}{2}$ and $M = D^{\pm 1/2}$, in which case $K$ is either the lazy random walk matrix $P$ or its transpose $P^T$. In these cases, the matrix $M$ encodes important geometric information about $G$, which motivates us to let

$$R_1 := M^{-1}M' \quad \text{and} \quad R_2 := M'M^{-1},$$

and consider the quantity

$$\kappa(G, G') := \max_{i=1,2} \{ \max\{ \|I - R_i\|_2, \|I - R_i^{-1}\|_2 \} \}$$

as a measure of how poorly aligned the degree vectors of $G$ and $G'$ are. In the general case, $\kappa(G, G')$ measures how different the $\| \cdot \|_M$ and $\| \cdot \|_{M'}$ norms are. It will also be useful to consider

$$R(G, G') := \max_{i=1,2} \{ \max\{ \|R_i\|_2, \|R_i^{-1}\|_2 \} \}$$

We note that by construction we have $1 \leq R(G, G) \leq \kappa(G, G') + 1$. Thus, if the norms $\| \cdot \|_M$ and $\| \cdot \|_{M'}$ are well-aligned, we will have $\kappa(G, G') \approx 0$ and consequently $R(G, G') \approx 1$. We note that we will have $\kappa(G, G') = 0$ and $R(G, G') = 1$, if either $M = I$ (so that $K = T$) or if $M = D^\pm 1/2$ and the graphs $G$ and $G'$ have the same degree vector. The latter situation occurs if e.g. $G$ is a regular graph and $G'$ is obtained by permuting the vertices of $G$. We also note that if $M$ is diagonal, e.g. if $M = D^{\pm 1/2}$, then $R_1 = R_2$.

We may also measure how far apart two graphs are via their spectral properties. In particular, if we let $V$ be the unitary matrix whose $i$-th column is given by $v_i$, an eigenvector or $T$ with eigenvalue $\lambda_i$, we see that two natural quantification’s of how poorly aligned the spectral properties of $G$ and $G'$ are given by

$$\max_{0 \leq i \leq n-1} |\lambda_i - \lambda'_i| \quad \text{and} \quad \| V - V' \|_2.$$

Motivated by e.g., [8], we also consider the “diffusion distances” given by

$$\| T - T' \|_M \quad \text{and} \quad \| K - K' \|_M.$$

### 4.1 Stability of the Wavelet Transforms

In this section, we analyze the stability of the wavelet transforms $W^{(1)}_j$ and $W^{(2)}_j$ constructed in Section 2. Our first two results provide a stability bounds for $W^{(1)}_j$ and $W^{(2)}_j$ in the case where $K = T$. These results will be extended to the general case by Theorem 9.

**Theorem 7.** Suppose $G = (V, E, W)$ and $G' = (V', E', W')$ are two graphs such that $|V| = |V'| = n$, and let $\lambda_1^* = \max\{ \lambda_1, \lambda'_1 \}$. Let $M = I$ so that $K = T$, let $W$ be the wavelets $W^{(1)}_j$ constructed from $T$ in Section 2 and let $W'$ be the corresponding wavelets constructed from $T'$. Then there exists a constant $C_{\lambda_1^*}$, depending only on $\lambda_1^*$ such that

$$\| (W - W') x \|_2^2 \leq C_{\lambda_1^*} \left( 2^J \sup_{1 \leq i \leq n-1} |\lambda_i - \lambda'_i|^2 + \| V - V' \|_2^2 \right),$$

where as in [2]

$$T = VAV^T \quad \text{and} \quad T' = V'AV'(V')^T.$$

**Proof.** By [8] and the fact that $q_j(t) = p_j(t)^{1/2}$, we have that for all $0 \leq j \leq J + 1$,

$$q_j(T) - q_j(T') = \frac{Vq_j(\lambda)VT - Vq_j(\lambda')(V')^T}{Vq_j(\lambda)VT - Vq_j(\lambda')(V')^T + (V - V')q_j(\lambda')(V')^T + V'q_j(\lambda')(V - V')^T}.$$

Therefore, since $V$ and $V'$ are unitary, we have that for all $x \in L^2(G)$

$$\| q_j(T)x - q_j(T')x \|_2 \leq \| (q_j(\lambda) - q_j(\lambda'))VT x \|_2 + \| V - V' \|_2 \| q_j(\lambda')VT x \|_2 + \| q_j(\lambda)(V - V')x \|_2.$$
and so summing over \( J \) yields
\[
\| (W-W')x \|_{L^2(G)}^2 \leq 3 \left( \sum_{j=0}^{j+1} \left( \|q_j(\Lambda) - q_j(\Lambda')\| V^T x \|_2^2 + \|V - V'\|_2^2 \|q_j(\Lambda') V^T x \|_2^2 \right) \right).
\]

For any sequence of diagonal matrices \( B_0, \ldots, B_{J+1} \) one has that for any \( y \in L^2(G) \)
\[
\sum_{j=0}^{J+1} \| B_j y \|_2^2 = \left\| \left( \sum_{j=0}^{J+1} B_j \right)^{1/2} y \right\|_2^2.
\]

Therefore, by \( \text{(6)} \),
\[
\sum_{j=0}^{J+1} \| q_j(\Lambda') V^T x \|_2^2 = \| V^T x \|_2^2 = \| x \|_2^2,
\]
and
\[
\sum_{j=0}^{J+1} \| q_j(\Lambda)(V - V')^T x \|_2^2 = \| (V - V')^T x \|_2^2 \leq \| V - V' \|_2^2 \| x \|_2^2.
\]

Now, since \( \| x \|_2 = 1 \) and \( A_0 = \lambda_0' = 1 \),
\[
\sum_{j=0}^{J+1} \| (q_j(\Lambda) - q_j(\Lambda')) x \|_2^2 \leq \sup_{0 \leq i \leq n-1} \sum_{j=0}^{J+1} \| q_j(\lambda_i) - q_j(\lambda_i') \|_2 \\
= \sup_{1 \leq i \leq n-1} \sum_{j=0}^{J+1} \| q_j(\lambda_i) - q_j(\lambda_i') \|_2 \\
\leq \sup_{1 \leq i \leq n-1} (\lambda_i - \lambda_i') \sum_{j=0}^{J+1} \sup_{0 \leq t \leq \lambda_i} \| q_j'(t) \|_2^2.
\]

When \( j = 0 \), we have
\[
|q_0'(t)| = \left| \frac{d}{dt} \sqrt{1 - t} \right| = \frac{1}{2 \sqrt{1 - t}} \leq C_{\lambda_i} \quad \text{for all } 0 \leq t \leq \lambda_i.
\]

Likewise, for \( j = J + 1 \), we have
\[
|q_{J+1}'(t)| = \left| \frac{d}{dt} t^{2^{j-1}} \right| \leq 2^{j-1} \quad \text{for all } 0 \leq t \leq \lambda_i.
\]

For \( 1 \leq j \leq J \), we may write \( q_j(t) = q_1(u_j(t)) \), where \( u_j(t) = t^{2^{j-1}} \), and use the fact that \( |u_j(t)| \leq 1 \) for all \( 0 \leq t \leq 1 \) to compute
\[
|q_j'(t)| = \left| q_1'(u_j(t)) u_j'(t) \right| \\
= \left| \frac{1 - 2u_j(t)}{2t - t^2} \right| \leq \frac{2^{j-1} t^{2^{j-1}}}{\sqrt{1-t}} \\
\leq C_{\lambda_i} 2^{j-1}
\]
for all \( 0 \leq t \leq \lambda_i \). Therefore,
\[
\sum_{j=0}^{J+1} \sup_{0 \leq t \leq \lambda_i} |q_j'(t)| \leq C_{\lambda_i} \left( 1 + 2^J + \sum_{k=1}^{J} 2^k \right) \leq C_{\lambda_i} 2^J.
\]
Our next result provides stability bounds for $\mathcal{W}_j^{(2)}$ in the case where $M = I$ (i.e. when $K = T$). We note that while $\mathcal{W}_j^{(1)}$ has the advantage of being a tight frame, $\mathcal{W}_j^{(2)}$ has stronger stability guarantees, which in particular are independent of $J$. Our proof, which is closely modeled after the proofs of Lemmas 5.1 and 5.2 in [5], is given in Appendix 5. Due to a small improvement in the derivation, our result appears in a slightly different form than the result stated there.

**Theorem 8.** Suppose $G = (V, E, W)$ and $G' = (V', E', W')$ are two graphs such that $|V| = |V'| = n$, and let $\lambda_j = \max\{\lambda_1, \lambda_j'\}$. Let $M = I$ so that $K = T$, let $\mathcal{W}$ be the wavelets $\mathcal{W}_j^{(2)}$ constructed from $T$ in Section 2 and let $W'$ be the corresponding wavelets constructed from $T'$. Then

$$\|W - W'|^2_{L^2(G)} \leq C\lambda_j^2 \left(\|T - T'|^2 + \|T - T'\|^2\right).$$

Theorems 7 and 5 show that the wavelets $\mathcal{W}_j^{(1)}$ and $\mathcal{W}_j^{(2)}$ are stable on $L^2(G)$ in the special case that $M = I$. Our next theorem extends this analysis to general $M$. More generally, it can be applied to any situation where $\{r_i(T)\}_{i \in I}$ and $\{r_i(T')\}_{i \in I}$ form frames on $L^2(G)$ and $L^2(G')$, $I$ is some indexing set, and each of the $r_i$ is a polynomial or square root of a polynomial.

**Theorem 9.** Suppose $G = (V, E, W)$ and $G' = (V', E', W')$ are two graphs such that $|V| = |V'| = n$, and let $M$ and $M'$ be invertible matrices. Let $I$ be an indexing set, and for $i \in I$, let $r_i(\cdot)$ be either a polynomial or the square root of a polynomial. Suppose that $W^T = \{r_i(T)\}_{i \in I}$ forms a frame analysis operator on $L^2(G)$ and that $W^{T'} = \{r_i(T')\}_{i \in I}$ forms a frame analysis operator on $L^2(G')$, and assume $B \leq 1$ in (7) for both $W^T$ and $W^{T'}$. Let $K = T^{-1}TM$ and let $W^K$ and $W^{K'}$ be the frames defined by $\{r_i(K)\}_{i \in I}$ and $\{r_i(K')\}_{i \in I}$. Then,

$$\|W^K - W^{K'}\|^2_{L^2(G,M)} \leq 6 \left(\|W^T - W^{T'}\|^2_{L^2(G)} + \kappa(G, G')^2(\kappa(G, G') + 1)^2\right).$$

**Proof.** Let $\|x\|_M = 1$, and let $y = Mx$. Note that $\|y\|_2 = \|Mx\|_2 = \|x\|_M = 1$. By Lemma 2 and by (5) we have that for all $i \in I$

$$r_i(K) = M^{-1}r_i(T)M \quad \text{and} \quad r_i(K') = (M')^{-1}r_i(T')M'.$$

Therefore,

$$\|(r_i(K) - r_i(K'))x\|_M = \|M(M^{-1}r_i(T)M - (M')^{-1}r_i(T')M')x\|_2$$

$$= \|r_i(T)Mx - M(M')^{-1}r_i(T')M'M^{-1}Mx\|_2$$

$$= \|r_i(T)y - R_2^{-1}r_i(T')y\|_2$$

$$\leq \|r_i(T)y - R_2^{-1}r_i(T')y\|_2 + \|R_2^{-1}r_i(T')y - R_2^{-1}r_i(T')R_2y\|_2$$

$$\leq \|r_i(T) - r_i(T')\|y\|_2 + \|(I - R_2^{-1})r_i(T')y\|_2 + \|R_2^{-1}\|_2 \|r_i(T')(I - R_2)y\|_2$$

$$\leq \|(r_i(T) - r_i(T'))y\|_2 + \kappa(G, G')\|r_i(T')y\|_2 + R(G, G')\|r_i(T')(I - R_2)y\|_2.$$

Therefore, squaring both sides, summing over $j$, and using the nonexpansiveness of $W^T$ and the fact that $\|y\|_2 = 1$, we have

$$\sum_{i \in I} \|(r_i(K) - r_i(K'))x\|^2_M$$

$$\leq 3 \left(\sum_{i \in I} \|(r_i(T) - r_i(T'))y\|^2_2 + \kappa(G, G')^2 \sum_{i \in I} \|r_i(T')y\|^2_2 + R(G, G')^2 \sum_{i \in I} \|r_i(T')(I - R_2)y\|_2^2\right)$$

$$\leq 3 \left(\|W^K - W^{K'}\|^2_{L^2(G,M)} + \kappa(G, G')^2 + R(G, G)^2\|I - R_2\|y\|_2^2\right)$$

$$\leq 3 \left(\|W^K - W^{K'}\|^2_{L^2(G,M)} + \kappa(G, G')^2 + R(G, G)\kappa(G, G')^2\right)$$

$$\leq 6 \left(\|W^K - W^{K'}\|^2_{L^2(G,M)} + \kappa(G, G')^2(\kappa(G, G') + 1)^2\right).$$

where the last inequality uses the fact that $R(G, G') \leq (\kappa(G, G') + 1)$. □
The following corollaries are immediate consequences of Theorem 9 and of Theorems 7 and 8.

**Corollary 1.** Suppose $G = (V, E, W)$ and $G' = (V', E', W')$ are two graphs such that $|V| = |V'| = n$, let $M$ and $M'$ be invertible matrices, and let $\lambda_1^* = \max\{\lambda_1, \lambda_1'^*\}$. Let $J \geq 0$, let $W$ be the wavelet transform $W_{j}^{(1)}$ constructed from $K$ in Section 2, and let $W'$ be the corresponding wavelet transform constructed from $K'$. Then,

$$\|W - W'\|_{L^2(G)}^2 \leq C_{\lambda_1^*} \left( 2^J \sup_{1 \leq i \leq n-1} |\lambda_i - \lambda_i'|^2 + \|W - W'\|_2^2 + \kappa(G, G')^2(\kappa(G, G') + 1)^2 \right).$$

**Corollary 2.** Suppose $G = (V, E, W)$ and $G' = (V', E', W')$ are two graphs such that $|V| = |V'| = n$, let $M$ and $M'$ be invertible matrices, and let $\lambda_1^* = \max\{\lambda_1, \lambda_1'^*\}$. Let $J \geq 0$, let $W$ be the wavelet transform $W_{j}^{(2)}$ constructed from $K$ in Section 2, and let $W'$ be the corresponding wavelet transform constructed from $K'$. Then,

$$\|W - W'\|_{L^2(G)}^2 \leq C_{\lambda_1^*} \left( \|T - T'\|_2^2 + \|T - T'\|_2 + \kappa(G, G')^2(\kappa(G, G') + 1)^2 \right).$$

One might also wish to replace Corollaries 1 and 2 with inequalities written in terms of $\|K - K'\|_M$ rather than $\|T - T'\|_2$. This can be done by the following proposition. Recall that we think of the two Hilbert spaces $L^2(G, M)$ and $L^2(G, M')$ as being well-aligned if $\kappa(G, G') \approx 0$ and $R(G, G') \approx 1$. In this case, the right-hand side of (10) is approximately $\|K - K'\|_M$.

**Proposition 3.**

$$\|T - T'\|_2 \leq \kappa(G, G') \left( 1 + R(G, G')^3 \right) + R(G, G)\|K - K'\|_M.$$

**Proof.** Let $\|x\|_2 = 1$. Then, since $T = MKM^{-1}$,

$$\|(T - T')x\|_2 = \|MKM^{-1}x - MK'M^{-1}x\|_2$$

$$= \|MKM^{-1}x - (MM^{-1})M'K'M^{-1}x\|_2$$

$$= \|MK - K'R_1\|_M\|M^{-1}x\|_M$$

$$\leq \|K - K'R_1\|_M\|M^{-1}x\|_M$$

(17)

since $\|M^{-1}x\|_M = \|x\|_2 = 1$. By the triangle inequality,

$$\|K - K'R_1\|_M \leq \|K - K'R_1\|_M + \|R_1^{-1}K - R_1^{-1}K'R_1\|_M$$

$$\leq \|K\|_M\|I - R_1^{-1}\|_M + \|R_1^{-1}\|_M\|K - K'R_1\|_M$$

$$\leq \|K\|_M\|I - R_1^{-1}\|_M + \|R_1^{-1}\|_M\|K - K'\|_M + \|R_1^{-1}\|_M\|K'(I - R_1)\|_M$$

$$\leq \|K\|_M\|I - R_1^{-1}\|_M + \|R_1^{-1}\|_M\|K - K'\|_M + \|R_1^{-1}\|_M\|K'\|_M\|I - R_1\|_M$$

$$\leq \kappa(G, G') + R(G, G')\|K - K'\|_M + R(G, G')R(G, G')^2\kappa(G, G')$$

$$= \kappa(G, G') \left( 1 + R(G, G')^3 \right) + R(G, G)\|K - K'\|_M.$$

where we used the facts that $\|I - R_1\|_M \leq \kappa(G, G')$, $\|R_1\|_M \leq R(G, G')$, $\|K\|_M = 1$, and $\|K'\|_M \leq \|R_1\|_2\|R^{-1}\|_2 \leq R(G, G')^2$. \hfill \Box

Our next theorem shows that if $G$ and $G'$ are well-aligned, then the upper frame bound for $W$ can be used to produce an upper frame bound for $W'$ on $L^2(G, M)$. This result will play a key role in proving the stability of the scattering transform.

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Theorem 10. Suppose \( G = (V, E, W) \) and \( G' = (V', E', W') \) are two graphs such that \( |V| = |V'| = n \), and let \( M \) and \( M' \) be invertible \( n \times n \) matrices. Let \( J := \{0, \ldots, J\} \) for some \( J \geq 0 \), let
\[
\mathcal{W} = \{\Psi_j, \Phi_j\}_{j \in J}
\]
be either of the wavelet transforms, \( \mathcal{W}^{(1)} \) or \( \mathcal{W}^{(2)} \), constructed in Section 2, and let \( W' \) be the corresponding wavelet transform constructed from \( K' \). Then \( W' \) is a bounded operator on \( L^2(G, M) \) and
\[
\sum_{j \in J} \|\Psi_j x\|_M^2 + \|\Phi_j x\|_M^2 \leq R(G, G')^4 \|x\|_M^2.
\]

Proof. By Lemma 2 we have that if \( r \) is either a polynomial, or the square root of a polynomial, then
\[
r(K') = (M')^{-1}r(T')M'.
\]
Therefore, again applying Lemma 2 we have
\[
\|r(K')x\|_M = \|M((M')^{-1}r(T')M')(M^{-1}M)\|_2 x_2 \leq \|R_2^{-1}r(T')R_2 M\|_2 \leq \|R_2^{-1}\|_2 \|r(T')\|_2 \|R_2\|_2 \|M\|_2 = R(G, G')^2 \|r(K')\|_M \|x\|_M.
\]
Since \( \Phi_j \) and each of the \( \Psi_j \) are either a polynomial in \( K \) or the square root of polynomial in \( K' \), the proof follows by observing
\[
\sum_{j \in J} \|\Psi_j x\|_M^2 + \|\Phi_j x\|_M^2 \leq R(G, G')^4 \left( \sum_{j \in J} \|\Psi_j x\|_M^2 + \|\Phi_j x\|_M^2 \right) \leq R(G, G')^4 \|x\|_M^2,
\]
with the last inequality following from the fact that \( B = 1 \) in (7) by Propositions 1 and 2.

4.2 Stability of the Scattering Transform

In this section, we will prove a stability result for the scattering transform. We will state and prove our result in a great degree of generality in order both to emphasize that the stability of the scattering transform is a consequence of the stability of the underlying frame and so that our result can be applied to other graph wavelet constructions. Towards this end, we will assume that \( G = (V, E, W) \) and \( G' = (V', E', W') \) are two-weighted graphs such that \( |V| = |V'| = n \), let \( M \) and \( M' \) be \( n \times n \) invertible matrices, and assume that \( \mathcal{W} = \{\Psi_j, \Phi_j\}_{j \in J} \) and \( \mathcal{W}' = \{\Psi'_j, \Phi'_j\}_{j \in J} \) are frames on \( L^2(G, M) \) and \( L^2(G', M') \) such that \( B \leq 1 \) in (7). If \( \Pi \) is a permutation, we will let \( \mathcal{W}'' = \Pi \mathcal{W}^\Pi \Pi^T = \{\Pi \Psi'_j \Pi^T, \Pi \Phi'_j \Pi^T\}_{j \in J} \) denote the corresponding permuted wavelet frame on \( G'' := \Pi G' \).

Our stability bound for the scattering transform will depend on choosing the optimal permutation \( \Pi \) such that \( \mathcal{W}'' = \Pi \mathcal{W}^\Pi \Pi^T \) is well-aligned with \( \mathcal{W} \) and has an upper frame bound on \( L^2(G, M) \) that is not too large. For \( \Pi \in S_n \), we let
\[
A_{\Pi}(G, G') := \sup_{\|x\|_M = 1} \|Wx - \Pi W' \Pi^T x\|_{L^2(G, M)}^2
\]
and
\[
C_{\Pi}(G, G') := \sup_{\|x\|_M = 1} \|\Pi W' \Pi^T x\|_{L^2(G, M)}^2.
\]
We will also let \( A(G, G') = A_1(G, G') \) and \( C(G, G') = C_1(G, G') \) when \( \Pi \) is the identity.

Theorem 11 provides stability guarantees for the windowed and non-windowed scattering transform with bounds that are functions of \( A(G, G') \) and \( C(G, G') \). Corollary 3 uses the permutation invariance results of Theorems 5 and 6 to extend these results by infimizing the same functions over all permutations. Since the
non-windowed scattering transform is always fully permutation invariant, this corollary will always apply to it. By Theorem 5, it will apply to the windowed scattering transform when \( M = D^{1/2} \) (or any other case in which the windowed scattering transform has provable invariance guarantees). These results imply, by Theorems 4, 5, 9, and 10 that the scattering transforms constructed from \( W_{j}^{(1)} \) or \( W_{j}^{(2)} \) are stable in the sense that the spectral properties of \( G \) are similar to the spectral properties of \( G' \) and the \( \| \cdot \|_M \) and \( \| \cdot \|_{M'} \) norms are well-aligned, then the scattering transforms \( S \) and \( S' \) will produce similar representations of an inputted signal \( x \). Many of the ideas in the proof of Theorem 11 are similar to those used to prove Theorem 5.3 in [8].

The primary difference is Lemma 4 which is needed because \( W' \) is a non-expansive frame on \( L^2(G, M') \), but not in general a non-expansive frame on \( L^2(G, M) \).

**Theorem 11.** Let \( G = (V, E, W) \) and \( G' = (V', E', W') \) be two graphs such that \( |V| = |V'| = n \), let \( M \) and \( M' \) be invertible \( n \times n \) matrices, and let \( J \) be an indexing set. Let \( W = \{ \Psi_j, \Phi \}_{j \in J} \) and be \( W' = \{ \Psi_j', \Phi' \}_{j \in J} \) be frames on \( L^2(G, M) \) and \( L^2(G', M') \), such that \( B \leq 1 \) in (7), and let \( \mu \) and \( \mu' \) be weighting vectors on \( L^2(G, M) \), and \( L^2(G', M') \). Let \( S_{\ell}^{(L)} \), \( (S_{\ell}^{(L)})' \), \( S_{\ell}' \), and \( (S_{\ell}')' \) be the partial windowed and non-windowed scattering transforms on \( G \) and \( G' \) with coefficients from layers \( \ell \leq m \leq L \). Then, for all \( x \in L^2(G, M) \)

\[
\left\| S_{\ell}^{(L)} x - (S_{\ell}^{(L)})' x \right\|_{L^2(L^2(G, M))} \leq \sqrt{2} \left( \sum_{m=\ell}^{L} \sum_{k=0}^{m} C_{\Pi}(G, G')^k / 2 \right) \| x \|_M, \quad (19)
\]

and

\[
\left\| S_{\ell}' x - (S_{\ell}')' x \right\|_{L^2(L^2(G, M))} \leq \sqrt{2} \left( (L - \ell) \| \mu - \mu' \|_M + \| \mu' \|_M \sqrt{A_{\Pi}(G, G')} \sum_{m=\ell}^{L} \sum_{k=0}^{m-1} C_{\Pi}(G, G')^k / 2 \right) \| x \|_M. \quad (20)
\]

**Corollary 3.** Under the assumptions of Theorem 11 the non-windowed scattering transform satisfies

\[
\left\| S_{\ell}^{(L)} x - (S_{\ell}^{(L)})' x \right\|_{L^2(L^2(G, M))} \leq \sqrt{2} \inf_{\Pi \in S_n} \left( (L - \ell) \| \mu - \mu' \|_M + \| \mu' \|_M \sqrt{A_{\Pi}(G, G')} \sum_{m=\ell}^{L} \sum_{k=0}^{m-1} C_{\Pi}(G, G')^k / 2 \right) \| x \|_M. \quad (21)
\]

Moreover, if we further assume that the windowed scattering transform \( (S_{\ell}^{(L)})' \) is permutation invariant up to a factor of \( B \) in the sense that for all \( \Pi \in S_n \) and for all \( x \in L^2(G, M) \),

\[
\left\| (S_{\ell}^{(L)})'' \Pi x - (S_{\ell}^{(L)})' x \right\|_{L^2(L^2(G, M))}^2 \leq B \| x \|_M. \quad (22)
\]

then

\[
\left\| S_{\ell}^{(L)} x - (S_{\ell}^{(L)})' x \right\|_{L^2(L^2(G, M))} \leq \left( B + \inf_{\Pi \in S_n} \sqrt{2} \| A_{\Pi}(G, G') \sum_{m=\ell}^{L} \sum_{k=0}^{m} C_{\Pi}(G, G')^k / 2 \right) \| x \|_M. \quad (23)
\]

**The Proof of Theorem 11.** Let \( A := A_{\Pi}(G, G') \) and \( C := C_{\Pi}(G, G') \). By the triangle inequality,

\[
\left\| S_{\ell}^{(L)} x - (S_{\ell}^{(L)})' x \right\|_{L^2(L^2(G, M))} \leq \left( \sum_{m=\ell}^{L} \sum_{j \in J^m} \| S_{\ell}^{(L)} x - S_{\ell}^{(L)}' x \|_M \right)^{1/2} \leq \left( \sum_{m=\ell}^{L} \sum_{j \in J^m} \| S_{\ell}^{(L)} x - S_{\ell}^{(L)}' x \|_M \right)^{1/2}.
\]
Therefore, to prove (19) it suffices to show

\[
\sum_{j \in J^m} \| S[j]x - S'[j]x \|^2 \leq 2A \cdot \left( \sum_{k=0}^{m} C^{k/2} \right)^2 \| x \|^2
\]  

(24)

for all \( 0 \leq m \leq L \). Similarly, to prove (20), it suffices to show

\[
\sum_{j \in J^m} \left\| \Phi S[j]x - \Phi S'[j]x \right\|^2 \leq 2\| \mu - \mu' \|^2 \| x \|^2 + 2\| \mu' \|^2 \| x \|^2 \leq A \cdot \left( \sum_{k=0}^{m-1} C^{k/2} \right)^2 \| x \|^2
\]  

(25)

for all \( 0 \leq m \leq L \), and then use the inequality \( \sqrt{a^2 + b^2} \leq |a| + |b| \).

Since the zeroth-order windowed scattering coefficient of \( x \) is given by

\[ S[j_e]x = \Phi x, \]

where \( j_e \) is the empty-index, we see that by the definition of \( A \) we have

\[
\sum_{j \in J^0} \| S[j]x - S'[j]x \|_M^2 = \| \Phi x - \Phi' x \|_M^2 \leq \| \mathcal{W} x - \mathcal{W}' x \|_{L^2(G,M)}^2 \leq A \| x \|_M^2.
\]

Therefore, (24) holds when \( m = 0 \). Similarly, since \( \overline{S[j_e]}x = \langle \mu, x \rangle_M \), we see that (25) holds when \( m = 0 \). The case where \( 1 \leq m \leq L \) relies on the following two lemmas. They iteratively apply the assumption that \( B \leq 1 \) in (7) and use the definitions of \( A \) and \( C \) to bound \( \{ U[j]x \}_{j \in J^m} \) and \( \left( \sum_{j \in J^m} \| U[j]x - U'[j]x \|_M^2 \right)^{1/2} \).

Full details are provided in Appendix F.

**Lemma 3.** For all \( m \geq 1 \),

\[
\sum_{j \in J^m} \| U[j]x \|_M^2 \leq \| x \|^2.
\]

**Lemma 4.** For all \( m \geq 1 \),

\[
\sum_{j \in J^m} \| U[j]x - U'[j]x \|_M^2 \leq A \left( \sum_{k=0}^{m-1} C^{k/2} \right)^2 \| x \|^2.
\]

For \( j \in J^m \), the triangle inequality implies that,

\[
\| S[j]x - S'[j]x \|_M = \| \Phi M \Psi_j \cdots M \Psi_{j_m} x - \Phi' M \Psi_j \cdots M \Psi_{j_m} x \|_M
\]

\[
\leq \left\| (\Phi - \Phi') M \Psi_j \cdots M \Psi_{j_m} x \|_M + \| \Phi' M \Psi_j \cdots M \Psi_{j_m} x - M \Psi_j \cdots M \Psi_{j_m} x \|_M
\]

\[
\leq \| \Phi - \Phi' \|_M \| M \Psi_j \cdots M \Psi_{j_m} x \|_M + \| \Phi' \|_M \| M \Psi_j \cdots M \Psi_{j_m} x - M \Psi_j \cdots M \Psi_{j_m} x \|_M.
\]

Therefore, by Lemmas 3 and 4

\[
\sum_{j \in J^m} \| S[j]x - S'[j]x \|_M^2 \leq 2\|| \Phi - \Phi' \|^2 \| M \Psi_j \cdots M \Psi_{j_m} \|_M^2
\]

\[
+ 2\| \Phi' \|^2 \| M \Psi_j \cdots M \Psi_{j_m} - M \Psi_j \cdots M \Psi_{j_m} x \|_M^2
\]

\[
\leq 2A \| x \|^2 + 2C \left( A^{1/2} \cdot \sum_{k=0}^{m-1} C^{k/2} \| x \|_M \right)^2
\]

\[
\leq 2A \cdot \left( \sum_{k=0}^{m} C^{k/2} \right)^2 \| x \|^2.
\]
which completes the proof of (24) and therefore of (19). Similarly, by the Cauchy-Schwarz inequality
\[
|S_j x - \bar{S}_j x| = |\mu M \Psi_{jm} \ldots M \Psi_{j1} x - \mu' M \Psi_{jm} \ldots M \Psi_{j1} x| \\
\leq |(\mu - \mu') M \Psi_{jm} \ldots M \Psi_{j1} x| + |\mu' (M \Psi_{jm} \ldots M \Psi_{j1} - M \Psi_{jm} \ldots M \Psi_{j1}) x| \\
\leq \|\mu - \mu'\|_M \|M \Psi_{jm} \ldots M \Psi_{j1} x\|_M + \|\mu'\|_M \|M \Psi_{jm} \ldots M \Psi_{j1} x - M \Psi_{jm} \ldots M \Psi_{j1} x\|_M.
\]

Squaring both sides and summing over \( j \) implies (25) and therefore (20).

The Proof of Corollary 3. Choose \( \Pi_0 \in S_n \) such that
\[
\sqrt{2A_{\Pi_0}(G, G')} \sum_{m=0}^{M} \sum_{k=0}^{m} \mathcal{C}_{\Pi_0}(G, G')^{k/2} = \inf_{\Pi \in S_n} \sqrt{2A_{\Pi}(G, G')} \sum_{m=0}^{M} \sum_{k=0}^{m} \mathcal{C}_{\Pi}(G, G')^{k/2}.
\]

Let \( G'' = \Pi_0 G' \), and let \( S'' \) be the scattering transform on \( G'' \) constructed from the wavelets \( W' : = \Pi_0 W \Pi_0^{T} \). Then under the assumption \( 22 \), we see
\[
\left\| S''_{\ell}(x) - \left(S_{\ell}^{(L)}\right)' x \right\|_{\ell^2(L^2(G, M))} \leq \left\| S_{\ell}(x) - \left(S_{\ell}^{(L)}\right)' x \right\|_{\ell^2(L^2(G, M))} + \left\| \left(S_{\ell}^{(L)}\right)' x - \left(S_{\ell}^{(L)}\right)' x \right\|_{\ell^2(L^2(G, M))} \\
\leq \left\| S_{\ell}(x) - \left(S_{\ell}^{(L)}\right)' x \right\|_{\ell^2(L^2(G, M))} + B\|x\|_M.
\]

(23) now follows from Theorem 11. The proof of (11) is similar, using the fact that the non-windowed scattering transform is always fully permutation invariant by Theorem 6.

5 Future Work

As alluded to in Section 1.2, we believe that our work opens up several new lines of inquiry for future research. Graph scattering transforms typically get numerical results which are good, but not quite state of the art in most situations. Our work has introduced a large class of scattering networks with provable guarantees. Therefore, one might attempt to learn the optimal choices of the matrix \( M \) and the spectral function \( g \) based on training data and produce a network which retains the invariance and stability properties of the scattering transform but has superior numerical performance. This would be an important step towards bridging the gap between theory and practice by producing an increasingly realistic model of graph neural networks with provable guarantees. Another possible extension would be to consider a construction similar to ours but which uses the spectral decomposition of the unnormalized graph Laplacian rather than the normalized Laplacian. Such a work would generalize 17 in a manner analogous to the way that this work generalizes 8 and 9. Lastly, particularly in the case where \( M \) is a function of \( D \), e.g. when \( K = P \), one might wish to study the behavior of the graph scattering transform on data-driven graphs obtained by subsampling a Riemannian manifold \( \mathcal{M} \). Such data-driven graphs typically arise in high-dimensional data analysis and in manifold learning. It can be shown that, under certain conditions, the normalized graph Laplacian of the data-driven graph converges pointwise 5 15 or in a spectral sense 2 3 6 14 16 to the Laplace Beltrami operator on \( \mathcal{M} \) as the number of samples tends to infinity. It would be interesting to see if one could use these results to study the convergence of the graph scattering transforms constructed here to the manifold scattering transform constructed in 13.

A Lemma 5

In this section, we state and prove the following lemma which is useful in the proof of Theorems 11 and 2.

Lemma 5. Assume \( B \leq 1 \) in (7). Then, for all \( m \geq 1 \) and for all \( x \in L^2(G, M) \)
\[
\sum_{j \in J^m} \|U_j x\|^2_M \geq \sum_{j \in J^{m+1}} \|U_j x\|^2_M + \sum_{j \in J^m} \|S_j x\|^2_M \tag{26}
\]
Since by assumption we have $B = 1$. Furthermore, for all $x, y \in \mathbb{L}^2(G, M)$,

$$\sum_{j \in \mathcal{J}^m} \|U[j]x - U[j]y\|_M^2 \geq \sum_{j \in \mathcal{J}^m} \|U[j]x - U[j]y\|_M^2 + \sum_{j \in \mathcal{J}^m} \|S[j]x - S[j]y\|_M^2.$$ (27)

The Proof of Lemma 2 Since by assumption we have $B \leq 1$, in (27) it follows that for all $j \in \mathcal{J}^m$ that

$$\|U[j]x - U[j]y\|_M^2 \geq \sum_{j_{m+1} \in \mathcal{J}} \|\Psi_{j_{m+1}}(U[j]x - U[j]y)\|_M^2 + \|\Phi(U[j]x - U[j]y)\|_M^2.$$ (26)

Therefore,

$$\sum_{j \in \mathcal{J}^m} \|U[j]x - U[j]y\|_M^2 \geq \sum_{j \in \mathcal{J}^m} \left( \sum_{j_{m+1} \in \mathcal{J}} \|\Psi_{j_{m+1}}(U[j]x - U[j]y)\|_M^2 + \|\Phi(U[j]x - U[j]y)\|_M^2 \right)$$ (28)

$$= \sum_{j \in \mathcal{J}^m} \left( \sum_{j_{m+1} \in \mathcal{J}} \|\Psi_{j_{m+1}}U[j]x - \Psi_{j_{m+1}}U[j]y\|_M^2 + \|\Phi(U[j]x - U[j]y)\|_M^2 \right)$$

$$\geq \sum_{j \in \mathcal{J}^m} \left( \sum_{j_{m+1} \in \mathcal{J}} \|M\Psi_{j_{m+1}}U[j]x - M\Psi_{j_{m+1}}U[j]y\|_M^2 + \|\Phi(U[j]x - U[j]y)\|_M^2 \right)$$ (29)

$$= \sum_{j \in \mathcal{J}^m} \left( \sum_{j_{m+1} \in \mathcal{J}} \|U[j_{m+1}]U[j]x - U[j_{m+1}]U[j]y\|_M^2 + \|\Phi(U[j]x - U[j]y)\|_M^2 \right)$$

$$= \sum_{j \in \mathcal{J}^m} \|U[j]x - U[j]y\|_M^2 + \sum_{j \in \mathcal{J}^m} \|S[j]x - S[j]y\|_M^2.$$ (28)

This completes the proof of (27). (26) follows from setting $y = 0$. Lastly, we observe that if that $A = B = 1$ in (7) and $y = 0$, we have equality in the inequalities (28) and (29).

**B The Proof of Theorem 1**

**Proof.** Applying Lemma 3 which is stated in Appendix A and recalling that $U[j_n]x = x$, we see

$$\|Sx - Sy\|_M^2 = \lim_{N \to \infty} \sum_{m=0}^{N} \sum_{j \in \mathcal{J}^m} \|S[j]x - S[j]y\|_M^2$$

$$\leq \lim_{N \to \infty} \sum_{m=0}^{N} \left( \sum_{j \in \mathcal{J}^m} \|U[j]x - U[j]y\|_M^2 - \sum_{j \in \mathcal{J}^m} \|U[j]x - U[j]y\|_M^2 \right)$$

$$\leq \|x - y\|_M^2 - \limsup_{N \to \infty} \sum_{j \in \mathcal{J}^{N+1}} \|U[j]x - U[j]y\|_M^2$$

$$\leq \|x - y\|_M^2.$$ (28)

This proves (3). Turning our attention to $\bar{S}$, we have that for any $j$, $\|\bar{S}[j]x - \bar{S}[j]y\| = \langle \mu, U[j]x - U[j]y \rangle_M$

$$\leq \|\mu\|_M \|U[j]x - U[j]y\|_M$$

$$\leq \|\mu\|_M \|\Phi^{-1}(S[j]x - S[j]y)\|_M$$

$$\leq \|\mu\|_M \|\Phi^{-1}\|_M \|S[j]x - S[j]y\|_M.$$
follows from squaring both sides, summing over \( j \), and applying (8).

C The Proof of Theorem 2

Proof. Let \( m \geq 1 \), let \( t = 2^{j-1} \) if \( W = W^{(1)} \), and let \( t = 2^j \) if \( W = W^{(2)} \), so that in either case \( \Phi = T^t \). Let \( x \in L^2(G, M) \), and let \( y = U[j]x \). (3) implies that for any \( z \in L^2(G, M) \)

\[
T^t z = \sum_{i=0}^{n-1} \lambda_i^l(v_i, z)_{l^2}v_i.
\]

Therefore, by Lemma 2 and the relationship \( B \in I \), we have that for all let \( m \), we have that for all \( x \),

\[
K^t y = M^{-1} T^t (My) = \sum_{i=0}^{n-1} \lambda_i^l(v_i, My)_{l^2} M^{-1} v_i = \sum_{i=0}^{n-1} \lambda_i^l(v_i, My)_{l^2} u_i.
\]

By Parseval’s identity, the fact that \( \{u_0, \ldots, u_{n-1}\} \) is an orthonormal basis for \( L^2(G, M) \), and the fact that \( \lambda_0 = 1 \), we have that for all let \( j \in \mathcal{F}^m \),

\[
\|S[j]x\|^2_M = \|K^t y\|^2_M = \left\| \sum_{i=0}^{n-1} \lambda_i^l(v_i, My)_{l^2} u_i \right\|^2_M = \sum_{i=0}^{n-1} \lambda_i^l(v_i, My)_{l^2}^2 \geq |\langle v_0, My \rangle|_2^2.
\]

Since \( v_0 = \frac{d^{l/2}}{\|d^{l/2}\|_2} = \frac{d^{l/2}}{\|d\|_1}, \)

\[
|\langle v_0, My \rangle|_2^2 \geq d_{\min} \frac{\|My\|_1}{\|d^{l/2}\|_2} \geq d_{\min} \frac{\|My\|_2}{\|d\|_1} \geq d_{\min} \frac{\|y\|_2}{\|d\|_1}.
\]

Summing over \( j \), this gives

\[
\sum_{j \in \mathcal{F}^m} \|S[j]x\|^2_M \geq d_{\min} \frac{\|d\|_1}{\|d\|_2} \sum_{j \in \mathcal{F}^m} \|U[j]x\|^2_M.
\]

Therefore, by Lemma 5 (stated in Appendix A) we see

\[
\sum_{j \in \mathcal{F}^m+1} \|U[j]x\|^2_M \leq \sum_{j \in \mathcal{F}^m} \|U[j]x\|^2_M - \sum_{j \in \mathcal{F}^m} \|S[j]x\|^2_M \leq \left(1 - d_{\min} \frac{\|d\|_1}{\|d\|_2}\right) \sum_{j \in \mathcal{F}^m} \|U[j]x\|^2_M.
\]

This proves (9). To prove (11), we note that since \( B \leq 1 \) in (7), we have

\[
\sum_{j \in \mathcal{F}} \|U[j]x\|^2_M \leq \|x\|^2_M.
\]

Therefore, (11) holds when \( m = 0 \). The result follows for \( m \geq 1 \) by iteratively applying (10).  

\( \square \)
D The Proof of Theorem 3

Proof. By Propositions 1 and 2 and by Lemma 5 we have that for all \( m \geq 0 \)

\[
\sum_{j \in J^m} \|S[j|x]\|_M^2 = \sum_{j \in J^{m+1}} \|U[j|x]\|_M^2 - \sum_{j \in J^m} \|U[j|x]\|_M^2.
\]

Therefore,

\[
\|Sx\|_{2^2(L^2(G,M))}^2 = \lim_{N \to \infty} \sum_{m=0}^N \left( \sum_{j \in J^{m+1}} \|U[j|x]\|_M^2 - \sum_{j \in J^m} \|U[j|x]\|_M^2 \right)
\]

\[
= \|x\|_M^2 - \lim_{N \to \infty} \left( \sum_{j \in J^N} \|U[j|x]\|_M^2 \right)
\]

\[
= \|x\|_M^2,
\]

where the last equality follows from Theorem 2. \( \square \)

E The Proof of Theorem 8

Proof. Let \( v := v_0 \) denote the lead eigenvector of \( T \). Since \( \lambda_1 < 1 \), we may write

\[
T = vv^T + \tilde{T},
\]

where

\[
\|\tilde{T}\|_2 = \lambda_1 < 1,
\]

and \( \tilde{T} v = 0 \). Since \( v_0, v_1, \ldots, v_{n-1} \) form an orthonormal basis for \( L^2(G) \), we have that

\[
T^k = vv^T + \tilde{T}^k
\]

for all \( k \geq 1 \). Therefore, we see that

\[
\Psi_j = \begin{cases} 
I - T = I - vv^T - \tilde{T} & \text{for } j = 0 \\
T^{2^j-1} - T^{2^j} = T^{2^j-1} - \tilde{T}^{2^j} & \text{for } 1 \leq j \leq J
\end{cases}
\]

and

\[
\Phi_j = T^{2^j} = vv^T + \tilde{T}^{2^j},
\]

with similar equations being valid for \( \Psi'_j \) and \( \Phi'_j \). Thus,

\[
\|W - W'\|_2^2 = \|\Phi_j - \Phi'_j\|_2^2 + \sum_{j=0}^J \|\Psi_j - \Psi'_j\|_2^2 
\]

\[
\leq 4 \left( \|vv^T - v'v'^T\|_2^2 + \sum_{j=0}^J \|\tilde{T}^{2^j} - \tilde{T'}^{2^j}\|_2^2 \right)
\]

(31)

The following lemma follows by imitating equation (23) of [8] and summing over \( j \).

Lemma 6.

\[
\sum_{j=0}^J \left\| T^{2^j} - (\tilde{T})^{2^j} \right\|_2^2 \leq C_M \| T - \tilde{T} \|_2^2.
\]
Proof. Let
\[ H_j(t) = \left((T + (1-t)T')\right)^{2^j}. \]
Then
\[ \left\| T^{2^j} - \left(T\right)^{2^j} \right\|_2 = \left\| H_j(1) - H_j(0) \right\|_2 \leq \int_0^1 \| g'_j(t) \|_2 dt \leq \sup_{0 \leq t \leq 1} \| H'_j(t) \|_2. \]
Since
\[ H'_j(t) = \sum_{\ell=0}^{2^j-1} \left((T' + (1-t)T'\right)\left( T - T'\right) \left(T' + (1-t)T\right)^{2^j-\ell-1}, \]
and \( \|T\|_2, \|T'\|_2 \leq \lambda_1^* \), this implies
\[ \|H'_j(t)\|_2 \leq 2^j \left(\lambda_1^*\right)^{2^j-1-2} \|T - T'\|_2. \]

Therefore,
\[
\sum_{j=0}^J \left\| T^{2^j} - \left(T\right)^{2^j} \right\|_2 \leq \|T - T'\|_2^2 \sum_{j=0}^\infty 2^{2j} \left(\lambda_1^*\right)^{2j+1-2} = C_\lambda \left\| T^{2^j} - \left(T\right)^{2^j} \right\|_2^2
\]
since \( \lambda_1^* < 1 \).

By the triangle inequality,
\[ \left\| T - T' \right\|_2 \leq \|T - T'\|_2 + \|v'v^T - v'v'^T\|_2. \]

Therefore, by (31), we have
\[ \|W - W'\|_2^2 \leq C_\lambda \left(\left\| T - T'\right\|_2 + \|v'v^T - v'v'^T\|_2^2) \right. \]
\[ \|v'v^T - v'v'^T\|_2^2 \]
To bound \( \|v'v^T - v'v'^T\|_2^2 \), we note that for all \( x \)
\[ \|v'v^T - v'v'^T\|_2 \leq \|(v - v')v^T x\|_2 + \|(v - v')v'^T x\|_2 \]
\[ \leq \|v\|_2 \|v - v'\|_2 \|x\|_2 + \|v - v'\|_2 \|v'\|_2 \|x\|_2 \]
\[ \leq 2 \|v - v'\|_2 \|x\|_2 \]
with the last equality following from the fact that \( \|v\|_2 = \|v'\|_2 = 1 \). Therefore, the proof is complete, pending the following lemma (a restatement of Lemma 5.2 of [3]) which bounds \( \|v - v'\|_2 \).

Lemma 7.
\[ \|v - v'\|_2^2 \leq 2 \left\| T - T' \right\|_2 \cdot \frac{1}{1 - \lambda_1^*}. \]

\[ \Box \]

The Proof of Lemma 7. Let \( \alpha = \langle v, v' \rangle_2 \). Since \( T = vv^T + T' \) and \( T' = v'(v')^T + T' \), with \( T v = \alpha T' v' = 0 \), we see
\[
(T - T')v = (vv^T - v'(v')^T) v + (T - T') v
= v - \alpha v' - T' v - \alpha T' v'
= (I - T') (v - \alpha v').
\]
Since \( \|I - T\|_2 = 1 - \lambda_1' \geq 1 - \lambda_1^* \), and \( \|v - v'\alpha\|_2 \geq 1 - \alpha \), this implies
\[
(1 - \alpha)(1 - \lambda_1^*) \leq \|I - T (v - v'\alpha)\|_2 \leq \|(T - T')v\|_2 \leq \|T - T'\|_2
\]
since \( \|v - v'\|^2 = 2(1 - \alpha) \). Therefore, we have
\[
\|v - v'\|^2 \leq 2 \frac{\|T - T'\|^2}{1 - \lambda_1^*}.
\]
\( \square \)

### F  The Proof of Lemmas 3 and 4

**The Proof of Lemma 3.** When \( m = 1 \), this follows immediately from the fact that we have assumed that \( B \leq 1 \) in (7) and the fact that \( M \) is nonexpansive. Now, suppose by induction that the result holds for \( m \). Then, calling that \( t_m := \left( \sum_{j \in J^m} \|U[j]x - U'[j]x\|_M^2 \right)^{1/2} \). Since the modulus operator is nonexpansive, the definition of \( A \) implies \( t_1 \leq A^{1/2} \|x\|_M \). Now, by induction, suppose the result holds for \( m \). Then, calling that \( U[j] = M\Psi_j \ldots M\Psi_{j_1} \), we have

\[
t_{m+1} = \left( \sum_{j \in J^{m+1}} \|M\Psi_{j_{m+1}} \ldots M\Psi_{j_1} x - M\Psi_{j_{m+1}}' \ldots M\Psi_{j_1}'x\|_M^2 \right)^{1/2}
\]
\[
\leq \left( \sum_{j \in J^{m+1}} \|\Psi_{j_{m+1}} - \Psi_{j_{m+1}}'\|_M \|M\Psi_{j_m} \ldots M\Psi_{j_1} x\|_M^2 \right)^{1/2}
\]
\[
+ \left( \sum_{j \in J^{m+1}} \|\Psi_{j_{m+1}}' \|_M \|M\Psi_{j_m} \ldots M\Psi_{j_1} x - M\Psi_{j_{m+1}}' \ldots M\Psi_{j_1}'x\|_M^2 \right)^{1/2}
\]
\[
\leq A^{1/2} \left( \sum_{j \in J^{m+1}} \|M\Psi_{j_m} \ldots M\Psi_{j_1} x\|_M^2 \right)^{1/2}
\]
\[
+ C^{1/2} \left( \sum_{j \in J^{m}} \|M\Psi_{j_m} \ldots M\Psi_{j_1} x - M\Psi_{j_m} \ldots M\Psi_{j_1}'x\|_M^2 \right)^{1/2}
\]
\[
\leq A^{1/2} \|x\|_M + t_m C^{1/2} \|x\|_M
\]

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by the definitions of $A$ and $C$ and by Lemma 3. By the inductive hypothesis, we have that

$$t_m \leq A^{1/2} \sum_{k=0}^{m-1} C^k \|x\|_M.$$

Therefore,

$$t_{m+1} \leq A^{1/2} \|x\|_M + A^{1/2} \sum_{k=0}^{m-1} C^{(k+1)/2} \|x\|_M = A^{1/2} \sum_{k=0}^{m} C^k \|x\|_M.$$

Squaring both sides completes the proof. \hfill \Box

References


