# Distinguished Lecture Series

# Assaf Naor on The Lipschitz Extension Problem

ON AUGUST 28, 29, AND 30, 2012, ASSAF NAOR OF THE Courant Institute for Mathematical Sciences at New York University visited the Fields Institute to give the Distinguished Lecture Series in conjunction with the one month Focus Program on Whitney Extension Problems. As part of the Whitney Extension program, Naor chose to talk about the *Lipschitz extension problem*. This problem is conceptually similar to the type of problems addressed in the Focus Program, but has several aspects that are different. Thus the lectures, which were interesting on their own merit, also served as a nice complement to the work presented during other parts of the Focus Program.

The essential ingredients of the Lipschitz extension problem are the following: two metrics spaces $(Y, d_Y)$ and $(Z, d_Z)$, a subset $X \subset Y$, a function $f: X \to Z$, and the Lipschitz constant:

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{d_Z(f(x), f(y))}{d_Y(x, y)}.$$

The goal is then the following:

<u>Goal:</u> Extend $f$ to a function $\tilde{f}: Y \to Z$ such that $\|\tilde{f}\|_{Lip} \leq K \|f\|_{Lip}$.

As is the case with many other problems in mathematics, the Lipschitz extension problem is simple to state, yet the search for solutions has yielded a wide range of deep mathematics.

Naor used the first lecture to give an overview of the problem and to broadly explain its development from the classical period into the current, modern period. As explained in his first lecture, the classical period goes back to the 1930's and centers around the so-called *isometric* extension property. This property deals with perfect extensions and asks: for which pairs of metric spaces $(Y, d_Y)$ and $(Z, d_Z)$ does $K = 1$, regardless of the subset $X$? This isometric extension property does not hold in general, although certain well known special cases exist, such as when $Y$ is arbitrary and $Z = \mathbb{R}$ (nonlinear Hahn-Banach theorem) or when $Y$ and $Z$ are Hilbert spaces
(Kirszbraun's Theorem).

The modern theory of Lipschitz extensions revolves around the *isomorphic* extension property, that is when $K > 1$, as well as the case when $K = \infty$. In the latter, one is interested in extensions from $n$ point subsets of $Y$ (for which $K$ is always finite), and determining the rate of growth of $K$ as $n \to \infty$. While the condition is weaker than that of the isometric extension property, the proofs can actually be harder and are more varied due to the nature of the solutions. Results of this type first appeared in 1983 with the work of Marcus and Pisier, and then soon after in 1984 the well known work of Johnson and Lindenstrauss (and the famous Johnson-Lindenstrauss Lemma). One of the philosophical points raised by Johnson and Lindenstrauss is to understand the connection between linear extension theorems (that is extensions of linear operators) and nonlinear extension theorems (that is Lipschitz extensions). In particular, do nonlinear analogues for the linear extension theorems exist?

Aside from the intrinsic interest of the Lipschitz extension problem, what makes it relevant to such a wide range of researchers is its connections with other areas of mathematics. During his first lecture, Naor highlighted several such examples, drawing connections to problems in graph theory and theoretical computer science, among others. It was intriguing to note, though, that his own stated personal interest is primarily drawn from the usefulness of such results in doing the "grunt work" of an analyst as a local-global tool.

In the second lecture, Naor started by building upon the ideas from the first lecture and introduced several tantalizing open problems whose simple statements belied the difficulties standing in the way of their solutions. He then introduced two new concepts: absolute Lipschitz extendability and random metric partitions. The latter leverages ideas from analysis combined with probability theory to achieve powerful theorems concerning the construction of Lipschitz extensions.

In the third and final lecture, Naor returned to the idea of developing nonlinear analogues for linear extension theorems, but in the context of a few deep results of a different flavor. He started by introducing the notions of type and cotype, which are important for geometry and analysis of Banach spaces. These in turn are key ingredients in the celebrated Maurey's Extension Theorem for bounded linear operators between Banach spaces (type applies to the domain, cotype to the range).

Naor then asked the following question: when working with metric spaces (as opposed to Banach spaces), are there analogues to type and cotype that yield a nonlinear version of Maurey's Extension Theorem for the Lipschitz extension problem? The notion of metric spaces with *Markov type*, introduced by Ball in 1992, proved to be the correct analogue to type. In the same paper Ball also introduced the notion of Markov cotype for normed spaces and obtained a partial generalization of Maurey's Extension Theorem, but this has since been superseded by the notion of *metric Markov cotype* introduced more recently by Mendel and Naor. After explaining and motivating these new concepts, Naor presented the full version of Ball's nonlinear extension theorem, which he proved with Mendel. The theorem states, modulo some technical details, that if the domain $(Y, d_Y)$ has Markov type $p$ and the range $(Z, d_Z)$ has metric Markov cotype $p$, then a Lipschitz function from a subset of $Y$ into $Z$ can be extended in the isomorphic sense. As more results on which spaces have these properties come in, the power of the theorem continues to grow. As explained by Naor, though, the current results already illustrate that this theorem encompasses nearly all known Lipschitz extension results. Given that metric Markov cotype is a relatively new concept, it seems likely that this theorem will continue to serve as a powerful tool for proving new results related to Lipschitz extensions.

*Matthew Hirn (Yale University)*